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STUDY OF RELATIVISTIC DOPPLER'S EFFECT

[1]Md. Shah Alam and [2]Md. Didar Chowdhury

[1]Department of Physics, Shahjalal University of Science and Technology, Sylhet, Bangladesh, and [2]Department of Physics, Govt. Alia Madrasah, Sylhet, Bangladesh

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Abstract: The Doppler's effect is well known. We have studied the relativistic Doppler's effect by the Special Lorentz transformations, Most General Lorentz transformations and Mixed Number Lorentz transformations. It is observed that the phenomenon of relativistic Doppler's effect is the same in the three cases, i.e. Special, Most General and Mixed Number Lorentz transformations.

Keywords: Doppler's effect, Relativistic Doppler's effect, Most General Lorentz transformations, Mixed number Lorentz transformations. PACS: 03.30, +p

Introduction

Mixed Number

Mixed number [1-5], α is the sum of a scalar x and a vector $\vec{A}$, i.e.$\alpha = x + \vec{A}$. It has satisfactory mathematical tools [2]. We have given below the summery of the mathematical tools of Mixed number algebra.

Two Mixed numbers α and β are equal if x=y and $\vec{A} = \vec{B}$. The addition of the two Mixed numbers can be written as

$$\alpha + \beta = (x+y)+(\vec{A} = \vec{B}) \qquad (1)$$

where $\alpha = x + \vec{A}$ and $\beta = y + \vec{B}$.

The Mixed number, $\alpha = x + \vec{A} = (x + A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k})$, where x, A_1, A_2, and A_3 are scalar values and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are basic or unit Mixed numbers. The properties of the basic Mixed numbers are:

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \text{ i.e. } i^2 = 1, j^2 = 1, k^2 = 1 \text{ and}$$

$$\alpha\beta = (x+\vec{A})\,(y+\vec{B}) = x\,y + \vec{A}\,.\,\vec{B} + x\,\vec{B} + y\,\vec{A} + i\,\vec{A}\times\vec{B}$$

where $i = \sqrt{(-1)}$.

These have the following multiplication Table:

	1	$\hat{i}$	$\hat{j}$	$\hat{k}$
1	1	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	$\hat{i}$	1	i$\hat{k}$	-i$\hat{j}$
$\hat{j}$	$\hat{j}$	-i$\hat{k}$	1	i$\hat{i}$
$\hat{k}$	$\hat{k}$	i$\hat{j}$	-i$\hat{i}$	1

The product of two the Mixed numbers α and β is defined as

$$\alpha\beta = (x+\vec{A})\,(y+\vec{B}) = x\,y + \vec{A}\,.\,\vec{B} + x\,\vec{B} + y\,\vec{A} + i \qquad (2)$$

where $\vec{A}.\vec{B}$ is the scalar product and $\vec{A}$ x $\vec{B}$ is the vector product of the vectors $\vec{A}$ and $\vec{B}$. The product of Mixed numbers is associative, i.e.

$$(\alpha\beta)\,\gamma = \alpha\,(\beta\,\gamma) \qquad (3)$$

where $\gamma = z + \vec{C}$ is another Mixed number. Taking x = y = 0, we get from equation (2)

E-mail: shahalam032003@ yahoo.com
didar_35@ easy.com

$$\vec{A} \otimes \vec{B} = \vec{A} \, . \, \vec{B} + i \; \vec{A} \times \vec{B} \, . \tag{4}$$

(the symbol ⊗ is chosen for Mixed product)

Special Lorentz Transformations

Let us consider the two inertial frames of reference S and *S'*, where the frame S is at rest and the frame *S'* is moving along X-axis with velocity *V* with respect to the frame S. We have assumed that the origins of both the frames S and *S'* coincides at the time *t*=*t'* =0. The relationship between the coordinates of the frames S *(x, y, z, t)* and *S'*, *(x', y', z', t')* which is called the Special Lorentz transformations, can be written as [6].

$$\begin{aligned} x' &= \gamma \, (x - Vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma \, (t - Vx) \end{aligned} \tag{5}$$

where $\gamma = \dfrac{1}{\sqrt{(1 - \dfrac{V^2}{c^2})}}$

and c = 1.

The inverse special Lorentz transformations can be written as [6]

$$\begin{aligned} x &= \gamma \, (x' + V t') \\ y &= y' \\ z &= z' \\ t &= \gamma \, (t' + V x'). \end{aligned} \tag{6}$$

Most General Lorentz Transformations

When the velocity $\vec{V}$ of the frame *S'* with respect to the frame S is not along X-axis, i.e the velocity $\vec{V}$ has three components V_x, V_y and V_z, then the relationship between the coordinates of the frames S and *S'*, which is called the Most General Lorentz transformations, can be written as [7]

$$\begin{aligned} \vec{r}\,' &= \vec{r} + \vec{V} \, [(\frac{\vec{r} \, . \, \vec{V}}{V^2}) (\gamma - 1) - t\gamma \,] \\ t' &= \gamma \, (t - \vec{r} \, . \, \vec{V} \,) \end{aligned} \tag{7}$$

where $\gamma = \dfrac{1}{\sqrt{(1 - \dfrac{V^2}{c^2})}}$ and c = 1.

The inverse Most General Lorentz transformations can be written as

$$\begin{aligned} \vec{r} &= \vec{r}\,' + \vec{V} \, [(\frac{\vec{r}\,' \, . \, \vec{V}}{V^2}) (\gamma - 1) + t' \gamma \,] \\ t &= \gamma \, (t' + \vec{r}\,' \, . \, \vec{V} \,). \end{aligned} \tag{8}$$

Mixed Number Lorentz Transformations

Let us consider that the velocity $\vec{V}$ of the frame *S'* with respect to the frame S is not along X-axis i.e. the velocity $\vec{V}$ has three components V_x, V_y and V_z. Using the Mixed number algebra, the relationship between the coordinates of the frames S and *S'*, which is called the Mixed Number Lorentz Transformations can be written as [8-10]

$$\begin{aligned} \vec{r}\,' &= \gamma \, (\vec{r} - t \, \vec{V} - \, i \, \vec{r} \times \vec{V} \,) \\ t' &= \gamma \, (t - \vec{r} \, . \, \vec{V} \,). \end{aligned} \tag{9}$$

where $\vec{r}$ and $\vec{r}\,'$ be the space part of the frames S and *S'*, respectively.

The inverse Mixed Number Lorentz transformations can be written as [8-10]

$$\begin{aligned} \vec{r} &= \gamma \, (\vec{r}\,' + t' \, \vec{V} + i \, \vec{r}\,' \times \vec{V}) \\ t &= \gamma \, (t' + \vec{r}\,' . \vec{V}). \end{aligned} \tag{10}$$

Relativistic Doppler's Effect

The Doppler's effect [11] is an apparent change in frequency of a wave that results from the motion of source, or observer, or both. The relativistic Doppler's effect [12] is the change in frequency and wave length of light, caused by the relative motion of the source and the observer such as in the regular Doppler's effect, when taking into account effects of the special theory of relativity. The relativistic Doppler's effect is different from the true (non-relativistic) Doppler's effect as the equations include the time dilation effect of the special relativity. They describe the total difference in the observed frequencies and possess the required Lorentz symmetry.

We can analyze the Doppler's effect [13] of light by considering a light source as a clock that ticks v_0 times per second and emits a wave of light with each tick. In the case of observer receding from the light source, the observer travels the distance Vt away from the source between the ticks, which means that the light wave from a given tick takes $\frac{Vt}{c}$ longer to reach him than the previous one. Hence the total time between the arrival of successive waves is

$$T = t + \frac{Vt}{c} = t_0 \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}$$

and the observed frequency is

$$\nu = \nu_0 \sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}}$$

In the case of the observer approaching the light source, the observed frequency is

$$\nu = \nu_0 \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}.$$

We want to study this phenomenon by the Special, Most General and Mixed Number Lorentz transformations.

Relativistic Doppler's effect in the special Lorentz Transformations

Let us consider the two frames S and *S'*, the latter moving with velocity $\vec{V}$ relative to the former along a positive X-axis. Let the transmitter and the receiver be situated at origins O and O*'* of the frames S and *S'*, respectively. Let the two light signals or pulses be transmitted at time t = 0 and t = T, T being the true period of light pulses. Let $\Delta t'$ be the interval between the reception of these pulses by the receiver in the frame *S'*. Obviously $\Delta t'$ is the proper time interval *T'* between these pulses (since the observer or the receiver is at rest in frame *S'*).

Using the inverse Lorentz transformations, we can write

$$\Delta x = \frac{V\,T'}{\sqrt{1-V^2}} \tag{11}$$

and $$\Delta t = \frac{T'}{\sqrt{1-V^2}} \tag{12}$$

but

$$\Delta t = T + \Delta x \tag{13}$$

where c = 1.

Using (11), (12) and (13) we can write [14]

$$\nu' = \nu \sqrt{\frac{(1-V)}{(1+V)}} \tag{14}$$

where *v* and *v'* be the actual and the observed frequencies of light pulses, respectively. Equation (14) is the Doppler's relativistic formula for light waves in the Special Lorentz transformations. In terms of the actual and apparent wave lengths *λ* and *λ'*, equation (14) can be written as

$$\lambda' = \lambda \sqrt{\frac{(1+V)}{(1-V)}}. \tag{15}$$

Relativistic Doppler's effect in the Most General Lorentz transformations

Let us consider the two frames S and *S'* where the relative velocity of the frame *S'* and the frame S is not parallel to the X-axis, i.e. the velocity $\vec{V}$ of the frame *S'* has three components V_x, V_y, and V_z. Let the transmitter and the receiver be situated at origins O and *O'* of the frames S and *S'*, respectively. Let the two light signals or pulses be transmitted at time t = 0 and t=T, T being the true period of light pulses. Let $\Delta t'$ be the interval between the reception of these pulses by the receiver in the frame *S'*. Obviously $\Delta t'$ is the proper time interval *T'* between these pulses (since the observer or the receiver is at rest in frame *S'*). Since observer continues to be at *O'* all the time, the distance $\Delta\vec{r}$ covered by him in the frame *S'* during the reception of two pulses is zero.

Using the space part of the inverse Most General Lotentz transformations [7] of equation (8), we can write

$$\Delta\vec{r} = \Delta\vec{r}' + \vec{V}\left[\left(\frac{\Delta\vec{r}'.\vec{V}}{V^2}\right)(\gamma - 1)\right. \tag{16}$$

Since $\Delta\vec{r} = 0$ and $\Delta t' = T'$, the equation (16) can be written as

$$\Delta\vec{r} = \vec{V}\, T'\gamma. \tag{17}$$

This equation (17) shows that the second pulse has to travel this extra distance $\Delta\vec{r}$ more than the first pulse in the frame S. Also using the time part of the inverse Most General Lorentz transformations [7] of equation (8), we have

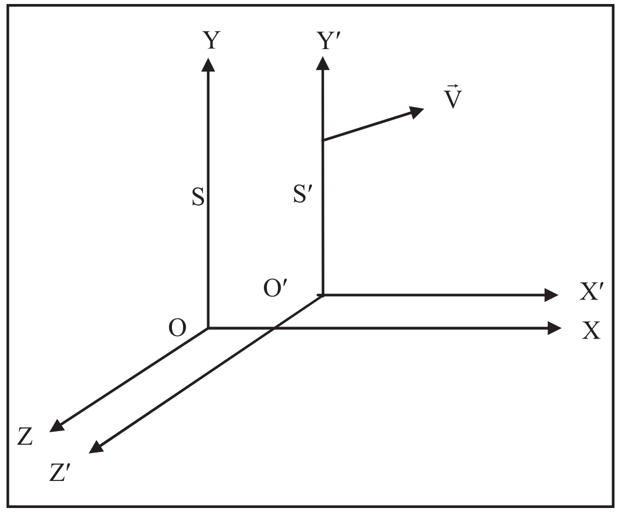

Fig 1. The frame *S'* is moving in any arbitrary direction with velocity $\vec{V}$ relative to the frame S.

$$\begin{aligned}\Delta t &= \gamma\,(\Delta t' + \Delta\vec{r}'.\vec{V}) \\ &= \gamma\, T'.\end{aligned} \tag{18}$$

(Since $\Delta\vec{r}' = 0$ and $\Delta t' = T'$)

This relation includes both the actual time period T of the pulses and the time taken $\left(\frac{\Delta\vec{r}}{c}\right)$ by the second pulse to cover the extra distance $\vec{r}$ in the frame S, i.e.

$$\begin{aligned}\Delta t &= T + \frac{\Delta\vec{r}}{c} \\ &= T + \Delta\vec{r}\end{aligned} \tag{19}$$

where c = 1.

Using equations (17), (18) and (19), we can write

$$T = T'\sqrt{\frac{(1-\vec{V})^2}{(1-V^2)}}. \tag{20}$$

If *ν* and *ν'* be the actual and observed frequencies of light pulses, respectively, we have $\nu = \frac{1}{T}$ and $\nu' = \frac{1}{T'}$, then equation (20) gives

$$\nu' = \nu\sqrt{\frac{(1-\vec{V})^2}{(1-V^2)}}. \tag{21}$$

This equation (21) is the formula of the relativistic Doppler's effect in the Most General Lorentz transformations for light. In terms of the actual and the apparent wave lengths, λ and λ', the equation (21) can be written as

$$\lambda' = \lambda \sqrt{\frac{(1-V^2)}{(1-\vec{V})^2}}. \tag{22}$$

Therefore, we can explain the phenomenon of relativistic Doppler's effect by the Most General Lorentz transformations.

If $(1+\vec{V})$ and $(1-\vec{V})$ be the two Mixed numbers, then using equation (2), we can write

$$\begin{aligned}(1+\vec{V})(1-\vec{V}) &= 1-\vec{V}.\vec{V}-\vec{V}+\vec{V}-i\,\vec{V}\times\vec{V} \\ &= 1-V^2.\end{aligned} \tag{23}$$

Using equation (23), equation (21) can be written as

$$\nu' = \nu \sqrt{\frac{(1-\vec{V})}{(1+\vec{V})}}. \tag{24}$$

Therefore, using the Mixed number algebra, the equation (24) is the formula of the relativistic Doppler's effect in the Most General Lorentz transformations.

Relativistic Doppler's effect in the Mixed number Lorentz transformations

Let us consider two frames S and *S'* where the velocity of the frame *S'* with respect to the frame S is not along X-axis, i.e. the velocity $\vec{V}$ of the frame *S'* has three components V_x, V_y and V_z. Let the transmitter and receiver be situated at origins O and *O'* of frames S and *S'*, respectively. Let the two light signals or pulses be transmitted at time t = 0 and t = T, T being the true period of light pulses. Let $\Delta t'$ be the interval between the reception of these pulses by the receiver in the frame *S'*. Obviously, since the observer or the receiver is at rest in frame *S'*, $\Delta t'$ is the proper time interval *T'* between these pulses. Since the observer continues to be at *O'* all the time, the distance $\Delta r'$ covered by him in the frame *S'* during the reception of the two pulses is zero.

Using the space part of the inverse Mixed Number Lorentz transformations [8-10] of equation (10), we can write

$$\Delta\vec{r} = \gamma\,(\,\Delta\vec{r}\,' + \Delta t'\,\vec{V} + i\,\Delta\vec{r}\,'\times\vec{V}\,) \tag{25}$$

Since $\Delta\vec{r}=0$ and $\Delta t'=T'$, then equation (25) can be written as

$$\begin{aligned}\Delta\vec{r} &= \gamma\,\Delta t'\,\vec{V} \\ or,\ \Delta\vec{r} &= \gamma\,T'\,\vec{V}\end{aligned} \tag{26}$$

This equation (26) shows that the second pulse has to travel this extra distance $\Delta\vec{r}$ than the first pulse in the frame S to be able to reach at origin *O'* in the moving frame *S'*. Using the time part of the inverse Mixed Number Lorentz transformations [8-10] of equation (10), we can write

$$\Delta t = \gamma\,(\,\Delta t' + \Delta\vec{r}\,'.\vec{V}\,). \tag{27}$$

Since $\Delta\vec{r}\,' = 0$ and $\Delta t' = T'$, then equation (27) can be written as

$$\Delta t = \lambda T'. \tag{28}$$

This relation includes both the actual time period T of the pulses and the time taken $(\frac{\Delta\vec{r}}{c})$ by the second pulse to cover the extra distance $\Delta\vec{r}$ in the frame S, i.e.

$$\Delta t = T + \frac{\Delta\vec{r}}{c}. \tag{29}$$

Using c = 1, the equation (29) can be written as

$$\Delta t' = \mathrm{T} + \Delta\vec{r}. \tag{30}$$

Using the equations (26), (28) and (30), we can write

$$T'\gamma = T + \vec{V}\, T'\, \gamma$$

$$or,\ T = T' \sqrt{\frac{(1-\vec{V})^2}{(1-V^2)}}. \quad (31)$$

Using equation (23), the equation (31) can be written as

$$T = T' \sqrt{\frac{(1-\vec{V})}{(1+\vec{V})}}. \quad (32)$$

If ν and ν' be the actual and observed frequencies of light pulses, respectively, we have $\nu = \frac{1}{T}$ and $\nu' = \frac{1}{T'}$, then equation (32) gives

$$\nu' = \nu \sqrt{\frac{(1-\vec{V})}{(1+\vec{V})}}. \quad (33)$$

This equation (33) is the formula of the relativistic Doppler's effect in the Mixed Number Lorentz transformations for light. In terms of the actual and the apparent wave lengths, λ and λ', the equation (33) can be written as

$$\lambda' = \lambda \sqrt{\frac{(1+\vec{V})}{(1-\vec{V})}}. \quad (34)$$

Therefore, we can explain the phenomenon of relativistic Doppler's effect by the Mixed Number Lorentz transformations.

Table 1
Comparison of the relativistic Doppler's effect in the special, Most General and Mixed Number Lorentz transformations.

Relativistic Doppler' effect	Special Lorentz transformations.	Most General Lorentz transformations.	Mixed Number Lorentz Transformations.
	$\nu' = \nu \sqrt{\frac{(1-V)}{(1+V)}}$	$\nu' = \nu \sqrt{\frac{(1-\vec{V})}{(1+\vec{V})}}$	$\nu' = \nu \sqrt{\frac{(1-\vec{V})}{(1+\vec{V})}}$

Conclusion

We have discussed the phenomenon of relativistic Doppler's effect by the Special, Most General and Mixed Number Lorentz transformations. Using the mixed number algebra, we have reformulated the formula of relativistic Doppler's effect in the Most General Lorentz transformations. We have observed that the formulae of relativistic Doppler's effect in the three cases, i.e. the Special, the Most General and the Mixed Number Lorentz transformations are the same. This paper will be helpful for the study of relativistic Doppler's effect in the case of three-dimensional velocity.

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