Proc. Pakistan Acad. Sci. 44(2):117-120.2007

MORE SOLITON SOLUTIONS TO THE KDV AND THE KDV-BURGERS EQUATIONS

Luwai Wazzan

Department of Mathematics, King Abdulaziz University, P.O.Box 80263, Jeddah 21589, Saudi Arabia

Received August 2006, accepted December 2006

Communicated by Prof. Dr. Q. K. Ghori

Abstract: In this work we use a generalized tanh method to solve the Korteweg-deVries equation and Korteweg-de Vries-Burgers equation. The main idea of the method is to take full advantage of the Riccati equation that the Tanh function satisfies. New multiple soliton solutions are obtained for the Korteweg-de Vries and Korteweg-de Vries-Burgers equations.

Keywords: Partial differential equations, generalized Tanh method, Riccati equation

Introduction

In recent years, finding exact solutions of nonlinear partial differential equations (PDE's) has become more attractive due to the use of widespread of computer algebraic systems (CAS) such as *Maple* and *Mathematica,* which allows us to do tedious and lengthy manipulations and help us to find new exact solutions of nonlinear PDE's. Many methods have been used to obtain traveling solitary wave solutions to non-linear PDE's, such as the inverse scattering method [1], Hirota's method [2], the hyperbolic tangent function method [3], the sine-cosine method [4], Backlund transformation method [5], the homogeneous balance [6,7], Darboux transformation [8], and the Jacobi elliptic function expansion method [9].

Recently, a generalized tanh function method [10] has been proposed to find new exact solutions to nonlinear PDE's. The goal of this work was to extend the generalized tanh function (GTF) method to solve the KdV and KdV-Burgers equations.

Description of the method

First, and before we describe the generalized tanh function method, we briefly discuss the tanh method.

Consider the general nonlinear wave PDE's, say, in two variables

$$u_t = G(u, u_x, u_{xx}, \ldots). \tag{1}$$

In order to apply the tanh method, the independent variables, x and t, are combined into a new variable, $\xi=\kappa(x-\omega t)$, where κ and ω represent the wave number and velocity of the traveling wave, respectively. Both are undetermined parameters with the assumption that $\kappa>0$. Therefore, $u(x,t)$ is replaced by $u(\xi)$, which defines the traveling wave solutions of (1). Equations such as Eq. (1) are then transformed into

$$-\kappa\omega\frac{du}{d\xi} = G\left(u, k\frac{du}{d\xi}, \kappa^2\frac{d^2u}{d\xi^2}, \ldots\right). \tag{2}$$

Hence, under the transformation $\xi=\kappa(x-\omega t)$, the PDE in Eq. (1) has been reduced to an ordinary differential equation (ODE) given by Eq. (2). The resulting ODE is then solved by a finite series of tanh functions of the form

$$u(\xi) = \sum_{j=0}^{n} a_j \tanh^j \xi, \tag{3}$$

where n is a positive constant that can be determined by balancing the linear term of highest order with the nonlinear term in (2), and $a_0,a,\ldots a_n$ are parameters to be determined.

The main idea of the generalized tanh function is to use the Riccati equation that the tanh function satisfies and replace the tanh ξ by the solution Y of the Riccati equation. The Riccati equation is given by

$$Y' = A + BY^2 \tag{4}$$

where $' := \frac{d}{d\xi}$ and A and B are constants. The following are special solution of Eq. (4).

Case 1: When A = ½, B = - ½ , then (4) has the solutions coth $\xi \pm$ csch ξ, tanh $\xi \pm$ i sech ξ($i^2 = -1$).

Case 2: When A = B =±½, then (4) has the solutions ξ sec $\xi \pm$ tan ξ, csc $\xi \pm i$cot ξ.

Case 3: When A = 1, B = -1, then (4) has the solutions tanh ξ, cot ξ.

Case 4: When A = B = 1, then (4) has the solutions tan ξ.

Case 5: When A = B = -1, then (4) has the solutions cot ξ.

By introducing the independent variable

Y = tanh ξ.

The solution of Eq. (1) can be written in the form

$$u(x,t) = u(\xi) = \sum_{j=0}^{n} a_j Y^j. \tag{5}$$

By balancing the linear term of highest order with the nonlinear term in Eq.(1), we can determine n in Eq. (5). Substituting (4) and (5) into (2) produces an algebraic equation in terms of $a_0,a_1,\ldots,a_n$. Since all coefficients of Y^j must vanish, $a_0,a_1,\ldots,a_n$. are determine d from the resulting relations.

To illustrate the method, we consider the KdV and the KdV-Burgers equation.

The KdV equation

Let us first consider the ID KdV equation which has the form

$$u_t + uu_x + \ u_{xxx} = 0 \tag{6}$$

$$u(x,t) = u(\xi), \qquad \xi = \kappa(x - \omega t). \tag{7}$$

Substituting (7) into (6), we obtain

$$-\kappa\omega u' + \kappa uu' + \varepsilon\kappa^3 u^{(3)} = 0 \tag{8}$$

Balancing the order of the nonlinear term uu' with the linear term $u^{(3)}$ in Eq. (8), we obtained $n = 2$. Thus, the solution has the form

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 \tag{9}$$

Substituting (4) and (9) into (8), collecting the coefficients of Y, yields to a system of algebraic equations. Solving the resulting system, with the aid of *Maple,* we obtain the following set of solutions:

$$a_0 = \omega - 8\varepsilon\kappa^2 AB, \quad a_1 = 0,$$
$$a_2 = -12\varepsilon\kappa^2 B^2, \ \kappa = \kappa, \quad \omega = \omega.$$

Using the special solutions of Eq. (4), we obtain

Case 1: When A = ½, B = - ½, then

$$a_0 = \omega + 2\varepsilon\kappa^2, \ a_1 = 0,$$
$$a_2 = -3\varepsilon\kappa^2, \ \kappa = \kappa, \ \omega = \omega,$$

and

$$u_1 = a_0 + a_2(\tanh\xi \pm \sec h\xi)^2,$$
$$u_2 = a_0 + a_2(\coth\xi \pm \csc h\xi),$$

Case 2: When A = B = ± ½, then

$$a_0 = \omega - 2\varepsilon\kappa^2, \; a_1 = 0,$$
$$a_2 = -3\varepsilon\kappa^2, \; \kappa = \kappa, \; \omega = \omega,$$

and

$$u_3 = a_0 + a_2(\sec\xi \pm \tan\xi)^2,$$

$$u_4 = a_0 + a_2(\csc\xi \pm \cot\xi)^2,$$

Case 3: When A = 1, B = -1, then

$$a_0 = \omega + 8\varepsilon\kappa^2, \quad a_1 = 0,$$
$$a_2 = -12\varepsilon\kappa^2, \quad \kappa = \kappa, \quad \omega = \omega,$$

and

$$u_5 = a_0 + a_2 \tanh^2 \xi,$$
$$u_6 = a_0 + a_2 \coth^2 \xi.$$

Case 4: When A = B = 1, then

$$a_0 = \omega + 8\varepsilon\kappa^2, \quad a_1 = 0,$$
$$a_2 = -12\varepsilon\kappa^2, \quad \kappa = \kappa, \quad \omega = \omega,$$

and

$$u_7 = a_0 + a_2 \tan^2 \xi.$$

Case 5: When A = B = -1, then

$$a_0 = \omega - 8\varepsilon\kappa^2, \; a_1 = 0,$$
$$a_2 = -12\varepsilon\kappa^2, \; \kappa = \kappa, \; \omega = \omega,$$

and

$$u_8 = a_0 + a_2 \cot^2 \xi,$$

where $\xi = \kappa(x - \omega t)$, and κ, ω are arbitrary constants.

The KdV-Burgers equation

$$u_t + uu_x - \nu u_{xx} + \varepsilon u_{xxx} = 0, \tag{10}$$

where ω and υ are positive constants.

In order to obtain traveling wave solutions for Eq. (10), we use

$$u(x,t) = u(\xi), \quad \xi = \kappa(x - \omega \mathrm{t}) \tag{11}$$

Substituting (11) into (10), gives

$$-\kappa\omega u' + \kappa uu' - \nu\kappa^2 u'' + \varepsilon\kappa^3 u''' = 0. \tag{12}$$

Balancing *uu'* with *uu'''* yields =2. Hence, Eq. (5) gives

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2. \tag{13}$$

Again substituting (4) and (13) into Eq. (12) and collecting the coefficients of Y, yields to a system of algebraic equations. Solving the resulting system, with the aid of *Maple,* we obtain the following sets of solutions:

$$a_0 = \frac{3\nu^2}{25\varepsilon} + \omega \;, \; a_1 = \pm\frac{6\nu^2 B}{25\varepsilon}\sqrt{\frac{-1}{AB}} \;,$$
$$a_2 = \frac{3B\nu^2}{25A\varepsilon}, \; \kappa = \pm\frac{\nu}{10\varepsilon}\sqrt{\frac{-1}{AB}} \;, \; \omega = \omega$$

Using the special solutions of Eq. (4), we obtain

Case 1: When A = ½, B = - ½, then

$$a_0 = \frac{3\nu^2}{25\varepsilon} + \omega \;, \; a_1 = \mp\frac{6\nu^2}{25\varepsilon},$$
$$a_2 = -\frac{3\nu^2}{25\varepsilon}, \kappa = \pm\frac{\nu}{5\varepsilon}, \omega = \omega,$$

and

$$u_1 = a_0 + a_1(\tanh\xi \pm i\sec h\xi) + a_2(\tanh\xi \pm i\sec h\xi)^2,$$
$$u_2 = a_0 + a_1(\coth\xi \pm \csc h\xi) + a_2(\coth\xi \pm \csc h\xi)^2.$$

Case 2: When A = B = ± ½, then

$$a_0 = \frac{3\nu^2 + 25\omega\varepsilon}{25\varepsilon},\; a_1 = \pm i\frac{6\nu^2}{25\varepsilon}, a_2 = \frac{3\nu^2}{25\varepsilon}, \kappa = \pm i\frac{\nu}{5\varepsilon}, \omega = \omega,$$

and

$$u_3 = a_0 + a_1(\sec\xi \pm \tan\xi) + a_2(\sec\xi \pm \tan\xi)^2,$$
$$u_4 = a_0 + a_1(\csc\xi \pm \cot\xi) + a_2(\csc\xi \pm \cot\xi)^2.$$

Case 3: When A = 1, B = -1, then

$$a_0 = \frac{3\nu^2 + 25\omega\varepsilon}{25\varepsilon}, a_1 = \mp\frac{6\nu^2}{25\varepsilon}, a_2 = -\frac{3\nu^2}{25\varepsilon}, \kappa = \pm\frac{\nu}{10\varepsilon}, \omega = \omega,$$

and

$$u_5 = a_0 + a_1\tanh\xi + a_2\tanh^2\xi,$$
$$u_6 = a_0 + a_1\coth\xi + a_2\coth^2\xi.$$

Case 4: When A = B = 1, then

$$a_0 = \frac{3\nu^2 + 25\omega\varepsilon}{25\varepsilon}, a_1 = i\frac{6\nu^2}{25\varepsilon}, a_2 = \frac{3\nu^2}{25\varepsilon}, \kappa = i\frac{\nu}{10\varepsilon}, \omega = \omega,$$

and

$$u_7 = a_0 + a_1\tan\xi + a_2\tan^2\xi.$$

Case 5: When A = B = -1, then

$$a_0 = \frac{3\nu^2 + 25\omega\varepsilon}{25\varepsilon}, a_1 = -i\frac{6\nu^2}{25\varepsilon}, a_2 = \frac{3\nu^2}{25\varepsilon}, \kappa = i\frac{\nu}{10\varepsilon}, \omega = \omega,$$

and

$$u_8 = a_0 + a_1\cot\xi + a_2\cot^2\xi,$$

where $\xi = \kappa\,(x - \omega t)$, and ω is an arbitrary constant.

Conclusion

In this article, the generalized tanh function method has been successfully implemented to find new traveling wave solutions for two nonlinear PDE's namely, the KdV and KdV-Burgers equations. The results show that the generalized tanh function method is a powerful Mathematical tool for obtaining exact solutions for the KdV and KdV-Burgers equations. It is also a promising method to solve other nonlinear partial differential equations.

References

1. **Gardner, C.S., Green, J.M., Kruskal, M.D. and Miura, R.M.** 1967. Method for solving the korteweg-de Vries equation. *Phys. Rev. Lett.* 19:1095-1097.
2. **Hirota, R.** 1971. Exact solution of Kortewege-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* 27:1192-1194.
3. **Malfeit, W.** 1992. Solitary wave solutions of nonlinear wave equations. *Am. J. Phys.* 60:650-654.
4. **Yan, C.T.** 1996. A simple transformation for nonlinear waves. *Phys. Lett.* A 244:77-84.
5. **Wadati, M.** 1973. The modified Kortweg-de Vries equation. *J. Phys. Soc. Jpn.* 34:1289-1296.
6. **Wang, M.L.** 1996. Exact solutions for compound KdV-Burgers equations. *Phys. Lett.* A 213:279-287.
7. **Yan, Z.Y. and Zhang, H.Q.** 2001. New explicit solitary solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water. *Phys. Lett.* A 285:355-362.
8. **Wadati, M. Sanuki, H. and Konno, K.** 1975. Relationships among inverse method, backlund transformation and an infinite number of conservation laws. *Prog. Theoret. Phys.* 53:419-436.
9. **Liu, S.K., Fu, Z.T., Liu, S.D. and Zhao, Q.** 2001. Expansion about the Jacobi elliptic function and its applications to nonlinear wave solutions. *Acta Phys. Sin.* 50:2068-2072.
10. **Chen, H. and Zhang, H.** 2004. New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto Sivanshinsky equation. *Choas, Soliton Fractals* 19:71-76.