ANALYTIC APPROXIMATIONS FOR VOLTERRA POPULATION EQUATION

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Abstract: In this paper,we give an analytic solution for the Volterra population equation using the homotopy analysis method and by the use of a fractional basis.

Keywords: Homotopy analysis method, Integro-differential equations

Introduction

Nonlinear equations describe most of phenomena in the real world. These equations are often handled by some techniques such as perturbation methods [1,2,3,4,5,6,7,8], nonperturbation methods and numerical methods. Perturbation techniques are based on the existence of small or large parameters in the investigated equation, and hence cannot deal with strongly nonlinear problems which have neither linear terms nor perturbation parameters. The nonperturbation methods such as the Adomian decomposition method [9,10] can treat strongly nonlinear problems but have some restrictions. The most evident restriction in the Adomian decomposition method is that the convergence region of its power solutions is generally small. In the numerical methods, numerical difficulties appear if a nonlinear problem has singularities or multiple solutions.

Liao [11] presented a powerful technique to obtain analytic approximate solutions for nonlinear problems in a more general flexible fashion. This technique is called the homotopy analysis method. He [11] applied this technique to give analytic approximate solution for the Volterra population equation [12], and by the use of an exponential basis. This equation is a nonlinear integro-differential equation of the form:

$$\beta \frac{d\,u(t)}{dt} = u(t) - u^2(t) - u(t)\int_0^t u(\tau)\,d\tau, \qquad (1)$$

subject to the intial condition

$$u(0) = \alpha, \qquad (2)$$

and was handled, also, by other techniques [13,14,15].

In equation (1), *u(t)* is the scaled population of individuals, t is the time, and *â=c /(a b)* is a nondimensional parameter such that $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient.

Here we apply the homotopy analysis method to obtain analytic approximate solution for equations {(1), (2)} but by the use of a fractional basis.

Application of the homotopy analysis method

Using a transformation of the form:

$$x = \lambda t, \; v(x) = u(t), \qquad (3)$$

where $\lambda > 0$ is a time - scale parameter , equation (1) takes the new form

$$\beta\lambda^2 \frac{d\,v(x)}{dx} = \lambda[v(x) - v^2(x)] - v(x)\int_0^x v(\tau)\,d\tau, \quad (4)$$

and with the initial condition

$$v(0) = \alpha, \quad (5)$$

We apply the homotopy analysis method on equations {(4), (5)} and then deduce the corresponding results for equations {(1) , (2)}, and in this application, we take the fractional basis

$$\{\frac{1}{(1+x)^k},\ k \ge 2\} \quad (6)$$

Also, we express the solution by the rule of expression

$$v(x) = \sum_{k=2}^{\infty} C_k \frac{1}{(1+x)^k},\ \{C_k \text{ is a coefficient}\}. \quad (7)$$

Viewing (4), (5), (6) and (7), we choose, respectively, the initial approximation and the auxiliary linear operator as follows:

$$v_0(x) = \alpha[\frac{1}{(1+x)^2}] + \gamma[\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}], \quad (8)$$

$$L = \frac{1}{2}(1+x)\frac{\partial}{\partial x} + 1, \quad (9)$$

where γ is a parameter to be determined later.

We construct the so-called zero-order deformation equation

$$(1-q)\,L[\phi(x,q) - v_0(x)] = q\,h\,H(x)\,N[\phi(x,q),\psi(q)], \quad (10)$$

subject to the initial condition

$$\phi(0,q) = \alpha, \quad (11)$$

where h is an auxiliary parameter, $H(x)$ is an auxiliary function and q is an embedding parameter varying from 0 to1. As q varies from 0 to 1, the solution of equation (10) varies from the initial approximation to the exact solution of equation (4).

Thus we have

$$\begin{aligned}&\phi(x,0) = v_0(x),\ \phi(x,1) = v(x),\ \psi(0) = \lambda_0\\ &= \text{initial value of } \lambda, \psi(1) = \lambda.\end{aligned} \quad (12)$$

The nonlinear operator N, in equation (10), is defined as

$$\begin{aligned}N[\phi(x,q),\psi(q)] &= \beta\psi^2(q)\frac{\partial\phi(x,q)}{\partial x} - \psi(q)[\phi(x,q) - \phi^2(x,q)]\\ &+ \phi(x,q)\int_0^x \phi(\tau,q)\,d\tau.\end{aligned} \quad (13)$$

By Taylor's series, at $q = 0$, we have

$$\phi(x,q) = \phi(x,0) + \sum_{m=1}^{\infty} v_m(x)\,q^m, \quad (14)$$

$$\psi(q) = \psi(0) + \sum_{m=1}^{\infty} \lambda_m\,q^m, \quad (15)$$

where

$$v_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x,q)}{\partial q^m}\Big|_{q=0}, \quad (16)$$

$$\lambda_m = \frac{1}{m!}\frac{\partial^m \psi(q)}{\partial q^m}\Big|_{q=0}. \quad (17)$$

If the parameters h, γ and the function $H(x)$ are properly chosen such that the series (14) and (15) converge at $q = 1$, then from (12)

$$v(x) = v_0(x) + \sum_{m=1}^{\infty} v_m(x), \quad (18)$$

$$\lambda = \lambda_0 + \sum_{m=1}^{\infty}\lambda_m. \quad (19)$$

Differentiating the zero-order deformation equation (10) m times with q and setting $q = 0$ and finally dividing it by $m!$, we obtain the m-th order deformation equation

$$L[v_m(x) - \chi_m v_{m-1}(x)] = h\,H(x)\,P_m(x),\ m \ge 1 \quad (20)$$

where

$$P_m(x) = \beta \sum_{n=0}^{m-1} v'_{m-1-n}(x) \sum_{i=0}^{n} \lambda_i \, \lambda_{n-i} - \sum_{n=0}^{m-1} \lambda_n \, v_{m-1-n}(x) + \sum_{n=0}^{m-1} \lambda_{m-1-n} \sum_{i=0}^{n} v_i(x) v_{n-i}(x) + \sum_{n=0}^{m-1} v_{m-1-n}(x) \int_0^x v_n(\tau)\, d\tau , \quad (21)$$

$$\chi_{\mathrm{m}} = \{ {}^{0\ ,\ \mathrm{m}=1}_{1\ ,\ \mathrm{m}\succ 1.} \quad (22)$$

The solutions of the quation (20) are referred as the *m*-th deformations, m≥1, and these deformations may satisfy the initial condition:

$v_m(0) = 0$, $m \geq 1$

It is natural for equations {(4),(5)} to take the intial value α to be small. Also, the auxiliary function $H(x)$ may be chosen such that there is no contradiction with the rule of expression (7), that is, no strange terms appear in the subsequent deformations from that of this rule.Also this choice may be consistent with the rule of coefficient ergodicity, that is, coefficients in (7) can be modified to ensure the completeness of the set of base functions. The solutions of equation (20) can be obtained by a symbolic software such as Mathematica, and it's solutions can be taken as:

$$v_m(x) = \sum_{k=2}^{2m+3} c_{m,k} \frac{1}{(1+x)^k} \text{ , } \{c_{m,k} \text{ is a coefficient}\}. \quad (23)$$

but since this equation contains the two unknowns: $\{ v_m(x), \lambda_{m-1},\ m \geq 1\}$,

we can use an additional algebraic equation in the second unknown to determine these unknowns successively.

For a best choice of *h*, *H(x)* and $\tilde{a}$ the *M*-th order approximation for the solution of equation {(4),(5)} is then

$$v(x) \approx v_0(x) + \sum_{m=1}^{M} \left(\sum_{K=2}^{2m+3} c_{m.k} \frac{1}{(1+x)^k} \right), \quad (24)$$

$$\lambda \approx \lambda_0 + \sum_{m=1}^{M} \lambda_m , \quad (25)$$

and consequently the *M*-th order approximation for the solution of equation {(1),(2)} takes the form

$$u(t) \approx \mathrm{v}_0(\lambda t) + \sum_{m=1}^{M} \left(\sum_{k=2}^{2m+3} c_{m,k} \frac{1}{(1+\lambda t)^k} \right). \quad (26)$$

The best values of both the parameters γ and *h* which control the convergence of the approximations of the kind (26) can be studied by investigating the convergence of the infinite integral $\int_0^\infty u(\tau)\, d\tau$.

This can be done by plotting the so-called h-curve which takes a horizontal line through the position of convergence.The convergence of this integral corresponding to some approximation of λ,with respect to *v (x)* implies it's convergence with respect to *u (t)* for this approximation of λ.

Results of the application

It is found that, for small positive values of $\gamma\,\beta$, the auxiliary function $H(x)$ can be chosen to take the value, $H(x) = 1$, to get best consistent analytic approximations for equations {(4), (5)} and hence for equations {(1), (2)}. $P_1(x)$, In the expression (21), and from (7) and (8), it is reasonable to represent $P_1(x)$, in the form

$$P_1(x) = \sum_{j=2}^{j_1} a_{1,j} \frac{1}{(1+x)^j}, \quad \{a_{1,j} \textit{ are coefficients}, j_1 \textit{ is an integer}\} \quad (27)$$

Since *H(x)* = 1, the right hand side of (21) contains the term $\frac{1}{(1+x)^2}$ and hence by means of the linear operator *L* in (9), the first deformation of (20) will contain a term of the form $\frac{1}{(1+x)^2} \ln(1+x)$.

Thus we have a contradiction with the rule of expression (7), and this implies that the coefficient in (27) at $j = 2$ may vanish which gives an algebraic equation from which we obtain

$$\lambda_0 = \frac{1}{2}\frac{2\alpha^2 + 3\alpha\gamma + \gamma^2}{\alpha + \gamma} \qquad (28)$$

In general, and by the same reasons the (m-1)-th deformation of λ can be determined from the equation

$$P_m(x) = \sum_{j=2}^{j_m} a_{m,j} \frac{1}{(1+x)^j}, \qquad (29)$$
$$\{a_{m,j} \text{ are coefficients}, \ j_m \text{ is an integer}\}$$

by equating the coefficient at j = 2 with zero.

Computing the approximations to Volterra population equation up to the third approximation and plotting the corresponding h-curve, it results that it is best to take h = -1, as shown in Figure 1 and Figure 2.

Figure 1 describes the h-curve

$$V = \int_0^\infty v(x)\,dx$$

for the initial α =1/10 and for γ < 1and {β < 1 or β > 1}: γ =1/2, β =1/5 (the thin line), γ =1/80, β = 8 (the solid line), γ =1/100, β =10 (the dashed line). In Figure 2, the h-curve of V is plotted for α =1/10, $\gamma \geq 1$ and β<1: γ =1, β =1/10 (the thin line), γ =2, β=1/20 (the solid line), γ=3, β =1/30 (the dashed line). The curve representing the solution of the Volterra population equation at h=-1, is plotted in Figures 3 and 4. In Figure 3: γ = 5, β = 0.1. Also, in Figure 4: γ =5, β = 0.1 (the thin line), γ = 4, β =0.2 (the solid line), γ = 3, β = 0.5 (the dashed line).

The convergent analytic results for the solution of the Volterra population equations {(1), (2)} are listed in Table 1. Figure 5 shows comparison with the Adomian method results of Wazwaz [14]. Finally, Figure 6 shows comparison with the numerical solution of TeBeest [15]. The two comparisons give the nearest results for the homotopy solution.

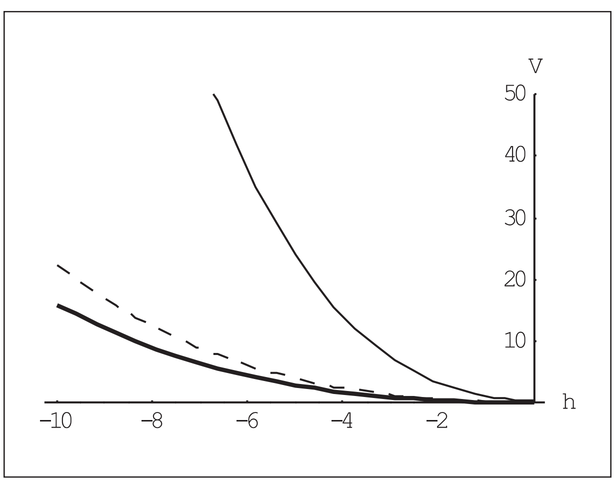

Figure 1: The h – curve of $V = \int_0^\infty v(x)\,dx$ for α = 1/10 and γ = 1/2, β = 1/5 (the thin line), γ = 1/80, β = 8 (the solid line) γ = 1/100, γ = 10 (the dashed line).

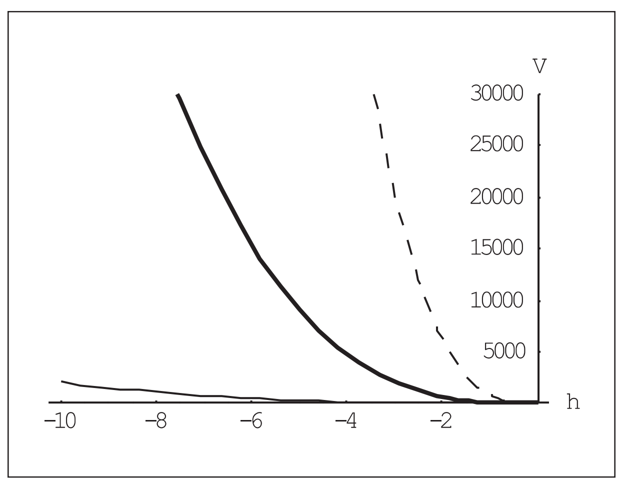

Figure 2: The h – curve of V for α = 1/10 and γ = 1, β = 1/10 (the thin line), γ = 2, β = 1/20 (the solid line) γ = 3, β = 1/30 (the dashed line).

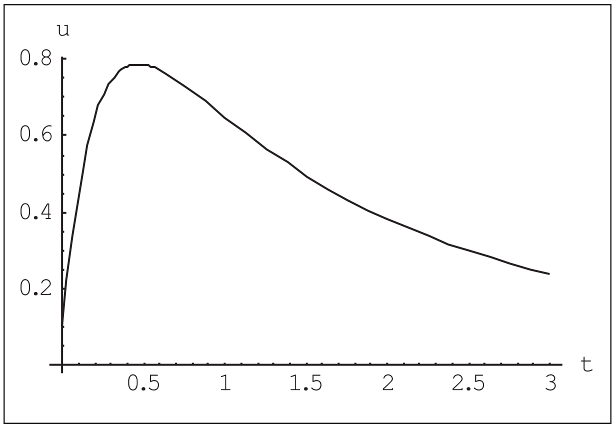

Figure 3: The solution of the Volterra population equation at h = – 1, β = 0.1, γ = 5.

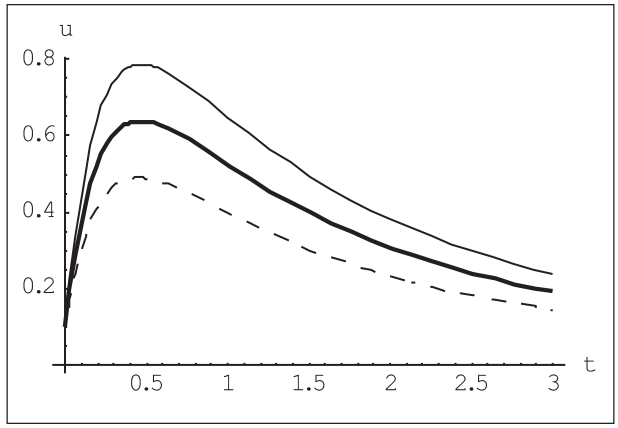

Figure 4: The solution of the Volterra population equation at h = – 1, β = 0.1, γ = 5 (the thin line), h = – 1, β = 0.2, γ = 4 (the solid line) and h = – 1, β = 0.5, γ = 3 (the dashed line).

Table 1

The analytic results for the solution of the Volterra population equation.

t	γ = 5 , β = 0.1	γ = 4 , β = 0.2	γ =3 , β = 0.5
0	0.1	0.1	0.1
0.2	0.648	0.532	0.417
0.4	0.779	0.634	0.488
0.6	0.771	0.625	0.479
0.8	0.717	0.579	0.442
1	0.65	0.525	0.4
2	0.381	0.307	0.233
3	0.241	0.194	0.147
4	0.164	0.132	0.1
5	0.119	0.095	0.072
6	0.089	0.072	0.055
7	0.069	0.056	0.043
8	0.056	0.045	0.034
9	0.046	0.037	0.028
10	0.038	0.031	0.023
20	0.011	0.009	0.007
30	0.005	0.004	0.003
50	0.002	0.002	0.001
100	0.0005	0.0004	0.0003

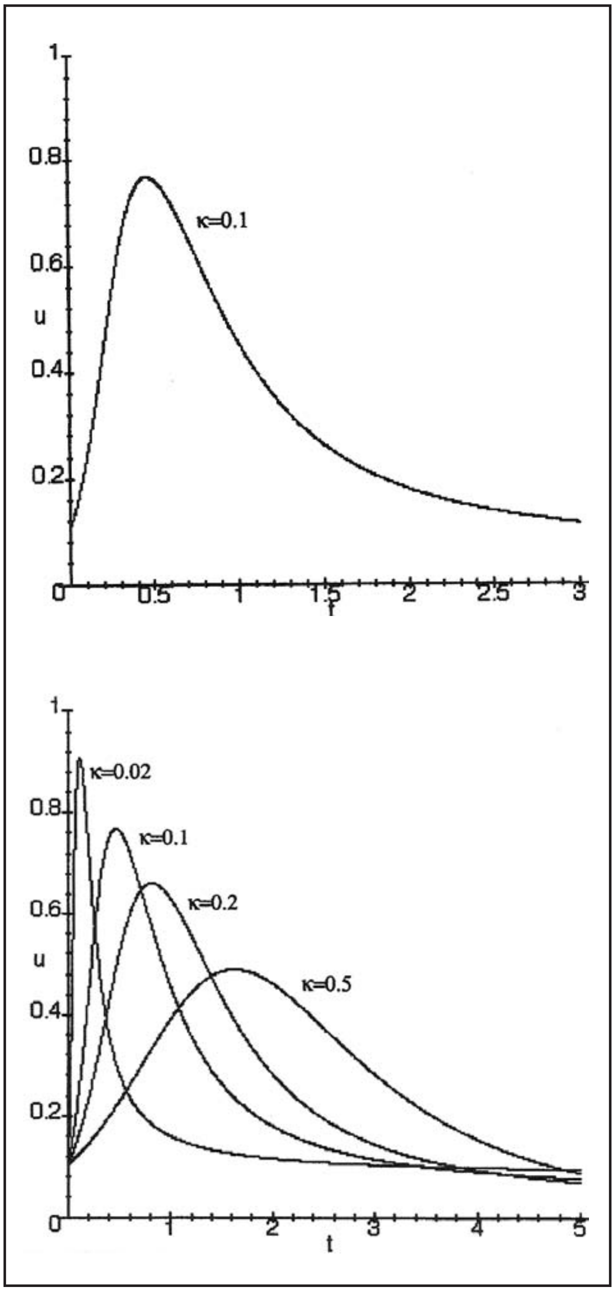

Figure 5: The Adomian method result of Wazwaz for population solution at $\alpha = 1/10$, $\kappa = \beta = 0.1$ and for different values of κ.

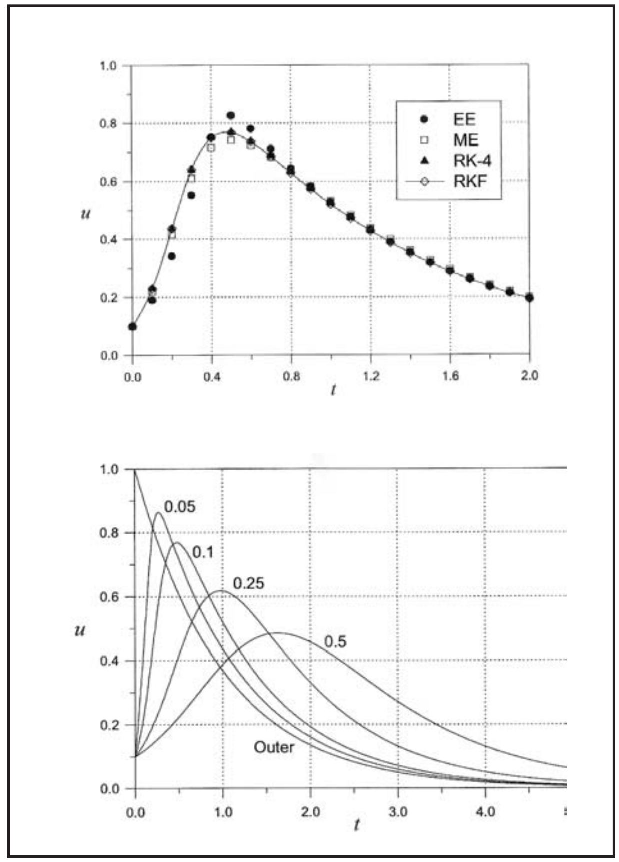

Figure 6: The numerical result of TeBeest for population solution at $\alpha = 1/10$, $\kappa = \beta = 0.1$ and for different values of κ.

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