

MODIFIED DIRAC EQUATION WITH CLASSICAL ZITTERBEWEGUNG

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Abstract: The integration between the special relativity theory and quantum mechanics yielded many paradoxes that remained unsolved until the last years, like the zitterbewegung problem. Despite the success of the Dirac equation, the spin prediction from it could be identified only with non-relativistic approximations (Pauli and Foldy-Wouthysen). In this paper, we show that the derivation of the spin and its magnetic moment can be done with a classical treatment. In this approach a modified Dirac equation was obtained which also eliminates the problem of the zitterbewegung.

Keywords: Dirac equation, electron spin, zitterbewegung.

Introduction

In 1926 while Schrödinger was publishing his non-relativistic single particle wave equation, Dirac [1] was searching for a relativistically invariant form of the one-particle Schrödinger equation for electrons starting from the relativistic equation, which was known as Klein–Gordon equation (KGE). However, at that time several objections emerged against the KGE as a single particle equation because its solutions allowed negative probability densities. Besides, there was the possibility of negative energies and their solutions did not have clear spin dependence. The concept of spin was introduced by Uhlenbeck and Goudsmit [2] and later formalized through the Pauli matrices and introduced into the non-relativistic Schrödinger equation. It was then apparent that it was not possible to introduce such spin matrices into the KGE. Because of this, KGE remained forgotten until 1934, when Pauli and Weisskopf partly re-established its validity by reinterpreting it as a field equation exactly as Maxwell’s equation for electromagnetic field and quantizing it.

Dirac published an equation in 1928 [1,3], which was presented as a definite solution to the above mentioned problems where he has shown that the spin belongs to the relativistic wave equation. The integration of the special relativity theory with quantum mechanics has yielded many paradoxes that remained unsolved until recent years. After the publication of the Dirac equation, a serious problem was discovered by Schrödinger [4]. This problem is known as “Zitterbewegung”. The second problem indeed was that it was impossible to directly write a non-relativistic equation for spin-1/2 particles. Instead, it could only be derived as a non-relativistic limit of the relativistic equation. Therefore, the Pauli equation for the theory of spin was derived as a non-relativistic limit of the relativistic equation and this has been known in standard quantum mechanics as a direct proof of the fundamentally relativistic nature of the spin [5]. However, this supposition was questioned by W. Greiner in 1984 [6] when he derived the spin from the non-relativistic quantum mechanics i.e. from the Schrödinger equation. Bakhoun [7] had

eliminated the problem of the “Zitterbewegung” by introducing a modification in the mass-energy equivalence principle. He introduced a new total relativistic energy formula $E = m v^2$ instead of Einstein’s $E = mc^2$, where m is the relativistic mass of the particle, and v is the particle velocity.

This paper carries Bakhoum’s work steps further, as we have previously derived Einstein’s equation $E = mc^2$ without using the special relativity theory; instead we started from classical physical laws like the Lorentz force law and Newton’s second law [8]. The energy formula of a particle $E = m v^2$ allows reconciliation between the de Broglie wave theory and the framework of the relativistic physics without the usual contradictions [9,10]. In this paper, using the new total relativistic energy formula $E = m v^2$, we have derived a modified Dirac equation and obtained the same result of Bakhoum concerning the “Zitterbewegung”. Furthermore, we reveal that the spin of the electron and its magnetic moment can be derived without using any kind of approximation.

The Relativistic Dirac Equation

The early twentieth century had witnessed two major revolutions in the way physicists understand the world. The first one was the theory of relativity and the other was quantum mechanics. Important results also emerged when these two theories were brought together; one of these results is the derivation of the electron spin that was known as a relativistic phenomenon.

When calculating kinetic energy relativistically using Lorentz transformation instead of Newtonian mechanics, Einstein discovered that the amount of the energy is directly proportional to the mass of a body.

$$E = mc^2 \quad (1)$$

The energy and momentum of a particle are then related by the principal equation governing the dynamics of a free particle.

$$E^2 = c^2 \mathbf{p}^2 + m_0^2 c^4. \quad (2)$$

Following Dirac, we take into account the time dependence as in the Schrödinger equation.

$$i\hbar \frac{\partial}{\partial t} \psi = \hat{H} \psi, \quad (3)$$

and assume that the energy operator \hat{H} can be expressed in terms of the momentum operator $\hat{\mathbf{p}}$ in a way as E is related to p in the non-relativistic case. Hence, using (2) and (3) we get

$$i\hbar \frac{\partial}{\partial t} \psi = \sqrt{c^2 \hat{\mathbf{p}}^2 + m_0^2 c^4} \psi.$$

One of the conditions imposed by Dirac in writing down a relativistic wave equation for the electron was that the “square” of that equation must give the KGE. Imposing the additional condition of linearity of the Hamiltonian \hat{H}_D in the components \hat{p}_k led Dirac to Eqs. (4) and (5),

$$i\hbar \frac{\partial}{\partial t} \psi = \hat{H}_D \psi, \quad (4)$$

$$\hat{H}_D = c(\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) + m_0 c^2 \beta. \quad (5)$$

where

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}; \quad k = 1, 2, 3, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad (6)$$

and σ_k are the 2 x 2 Pauli matrices, and I_2 is the identity matrix.

The non-relativistic limit of The Dirac equation.

In standard quantum mechanics, it is not possible to directly extend the Schrödinger equa-

tion to spinors, so the Pauli equation must be derived from the Dirac equation by taking its non-relativistic limit. This is in particular the case for the Pauli equation which predicts the existence of an intrinsic magnetic moment for the electron and gives its correct value only when it is obtained as the non-relativistic limit of the Dirac equation.

The Dirac equation for relativistic charged particle moving in a constant magnetic field could be written as follows

$$i\hbar \frac{\partial}{\partial t} \psi = [c\alpha \cdot (\hat{\mathbf{p}} - \frac{e}{c} \hat{\mathbf{A}}) + m_0 c^2 \beta] \psi. \quad (7)$$

We can derive the Pauli equation following the standard method of eliminating small components. We consider the two-component representation, where the four-component spinor ψ is decomposed into two two-component spinors ψ_b and ψ_s .

$$\psi = \begin{pmatrix} \psi_b \\ \psi_s \end{pmatrix}; \quad \psi_b = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_s = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \quad (8)$$

In the non-relativistic limit, the rest energy $m_0 c^2$ becomes dominant. Therefore, the two-component solution is approximately.

$$\psi_{b,s} = e^{\frac{-im_0 c^2 t}{\hbar}} \psi_{b,s}^0. \quad (9)$$

Substituting the non-relativistic solution in Eq.(7), in the Dirac representation, gives these two equations

$$i\hbar \frac{\partial}{\partial t} \psi_b^0 = c \cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) \psi_s^0, \quad (10a)$$

$$i\hbar \frac{\partial}{\partial t} \psi_s^0 = c \cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) \psi_b^0 - 2m_0 c^2 \psi_s^0. \quad (10b)$$

When the kinetic energy is small compared to the rest energy, ψ_s^0 will vary slowly as a function of time, so

$$\left| i\hbar \frac{\partial}{\partial t} \psi_s^0 \right| \ll |m_0 c^2 \psi_s^0|. \quad (11)$$

With this last approximation, Eq.(10b) becomes

$$0 = c \cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) \psi_b^0 - 2m_0 c^2 \psi_s^0.$$

This gives

$$\psi_s^0 = \frac{\cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}})}{2m_0 c} \psi_b^0 \quad (12)$$

The lower component ψ_s^0 is generally referred to as the 'small' component of the wavefunction relative to the 'large' component ψ_b^0 . Substituting the expression ψ_s^0 given by Eq. (12) into Eq. (10a) we obtain

$$i\hbar \frac{\partial}{\partial t} \psi_b^0 = \frac{1}{2m_0} \left(\cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) \right) \left(\cdot (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) \right) \psi_b^0.$$

Finally, by using the well-known identity

$$(\cdot \mathbf{a})(\cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i \cdot (\mathbf{a} \times \mathbf{b}),$$

we obtain, $\mathbf{B} = \nabla \times \mathbf{A}$ being the magnetic field,

$$i\hbar \frac{\partial}{\partial t} \psi_b^0 = \left[\frac{1}{2m_0} (i\hbar \nabla - \frac{e}{c} \hat{\mathbf{A}}) - \frac{e\hbar}{2m_0 c} \cdot \mathbf{B} \right] \psi_b^0. \quad (13)$$

We recognize here the Pauli equation for the theory of spin in non-relativistic quantum mechanics. As it is well known, one of the main results of the Pauli equation (when it is derived from the Dirac equation) is to yield the correct gyromagnetic factor $g = 2$ for the electron.

It is well known that it was indeed impossible to directly write a non-relativistic equation for spin-1/2 particles, and that it could therefore only be derived as a non-relativistic limit of the relativistic equation. Therefore, the Pauli equation for the theory of spin was derived as a non-relativistic limit of the relativistic equation and it was well known in standard quantum mechanics as a direct proof of the fundamentally relativistic nature of the spin.

Derivation of modified Dirac equation and its solutions

In several recent papers we suggested another way to account for the LT and its kinematical effects in relativistic electrodynamics as well as in relativistic mechanics. And by following the same approach we derived Einstein's equation $E = mc^2$ from classical physical laws such as the Lorentz force law and Newton's second law.

Let us consider two inertial systems S and S' with a relative velocity u // ox between them and consider from S a charged particle which has a momentum $\mathbf{p} = m\mathbf{v}$. So following a similar approach, we can obtain the relativistic Lorentz factor γ and the relativistic mass

$$m = \gamma m_0 \quad (14a)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (14b)$$

To obtain the same relativistic combination between the momentum and the energy of the charged particle, we start from (14b) and we write this equation as

$$\gamma^2 - \frac{u^2}{c^2} \gamma^2 = 1.$$

Multiplying both sides with $m_0^2 c^4$ we obtain

$$c^4 \gamma^2 m_0^2 - c^2 \gamma^2 m_0^2 u^2 = m_0^2 c^4 \quad (15)$$

We recognize that the term $\gamma^2 m_0^2 u^2$ represents the square of the momentum in the system S ($\mathbf{p}^2 = m^2 v^2$), and the root of the first term in Eq. (15) is

$$m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \gamma m_0 c^2 = mc^2. \quad (16)$$

mc^2 in Eq.(16) represents the total relativistic energy E . With this notation, Eq. (15) becomes

$$E^2 = c^2 \mathbf{p}^2 + m_0^2 c^4. \quad (17)$$

Eq. (17) represents the combination of the momentum and the energy of the same particle.

We obtained also the new energy formula

$$H = m v^2; \quad H \equiv E. \quad (18)$$

Eq. (18) would imply

$$H^2 = v^2 \mathbf{p}^2, \quad (19)$$

and since $1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}$, then Eq. (19) becomes

$$H^2 = c^2 \mathbf{p}^2 \left(1 - \frac{1}{\gamma^2}\right) = c^2 \mathbf{p}^2 - c^2 m_0^2 v^2. \quad (20)$$

Eq. (20) can be written in four dimensional formulation when we define the 4-d momentum vector as

$$p_\mu = (m \mathbf{v}, p_4) = \left(\mathbf{p}, \frac{iH}{c}\right); \quad H = m v^2. \quad (21)$$

Squaring Eq. (21), we get

$$p_\mu p^\mu = \mathbf{p}^2 - \frac{H^2}{c^2}. \quad (22)$$

From another side we have

$$p_\mu = m_0 u_\mu \text{ and } p_\mu p^\mu = m_0^2 u_\mu u^\mu = m_0^2 v^2.$$

The relationship between the equation $H = mv^2$ and the 4-vector (Minkowski) representation of special relativity is discussed in details by Bakhom [11], where the 4-velocity u_μ of the particle becomes $u_\mu = (\gamma v_x, \gamma v_x, \gamma v_x, i \gamma \frac{v^2}{c})$, and hence $u_\mu u^\mu = v^2$. So, we can replace the quantity $p_\mu p^\mu$ in Eq. (22) with $m_0^2 v^2$

$$m_0^2 v^2 = \mathbf{p}^2 - \frac{H^2}{c^2}.$$

This equation would imply

$$H^2 = c^2 \mathbf{p}^2 - m_0^2 c^2 v^2. \quad (23)$$

So, Eq. (23) is similar to Eq. (20). And now we can write the modified Dirac Hamiltonian from Eq. (19) by following the method of Dirac as

$$H = v \boldsymbol{\alpha} \cdot \mathbf{p} + v \beta p_4, \quad (24)$$

where the matrices α and β are the same as in Eq. (6).

Replacing H with the operator $i \hbar \frac{\partial}{\partial t}$ and \mathbf{p} with the operator $-i \hbar \nabla$ in Eq. (24), then modified Dirac equation can be written as follows

$$i \hbar \frac{\partial}{\partial t} \psi = -i \hbar v \alpha_j \nabla^j \psi + v p_4 \alpha_4 \psi. \quad (25)$$

We will continue to find the eigenfunctions of the new Hamiltonian for free electron to show that there is no contradiction between our results and the eigenfunctions problem for the Hamiltonian in Eqs. (4, 5) and to prove that there is no contradiction with Dirac's conceptions if c in Eq. (5) is exchanged with v in Eq.(24).

Solutions to Eq. (25) are plane waves which can be written in the following form

$$\Psi(x, t) = N \begin{pmatrix} \varphi(x, t) \\ \chi(x, t) \end{pmatrix} = N \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} e^{\frac{-iE_v t}{\hbar}}, \quad (26)$$

where N is the normalization constant, and $\begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}$ is a four component spinor, and $E_v = mv^2$.

Substituting Eq. (26) in Eq. (25), and considering that

$$\begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} e^{\frac{i p \cdot x}{\hbar}}, \text{ where } \varphi_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \chi_0 = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \text{ are two-component spinors, we find}$$

$$E_v \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} = v \begin{pmatrix} 0 & p_j \sigma^j \\ p_j \sigma^j & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} + v p_4 \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix}. \quad (27)$$

From Eq. (27) we obtain these two equations

$$\left. \begin{aligned} (E_v - v p_4) \varphi_0 - v p_j \sigma^j \chi_0 &= 0 \\ -v p_j \sigma^j \varphi_0 + (E_v + v p_4) \chi_0 &= 0 \end{aligned} \right\}. \quad (28)$$

These two equations have solution if

$$\begin{vmatrix} E_v - v p_4 & -v p_j \sigma^j \\ -v p_j \sigma^j & E_v + v p_4 \end{vmatrix} = 0,$$

or equivalently

$$E_v^2 = \mathbf{p}^2 + (v p_4)^2. \quad (29)$$

From Eqs. (28) we find

$$\chi_0 = \frac{v p_j \cdot \sigma^j}{E_v + v p_4} \varphi_0 \text{ and } \varphi_0 = \frac{v p_j \cdot \sigma^j}{E_v - v p_4} \chi_0. \quad (30)$$

Hence

$$\chi_0 = \frac{(v p_j \cdot \sigma^j)^2}{E_v^2 - (v p_4)^2} \chi_0. \quad (31)$$

Using Eq. (29) in Eq. (31), we find that χ_0 can take one of these two representations

$$\chi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From the above calculations we find

$$\begin{aligned} \Psi(x, t) &= N \begin{pmatrix} \varphi(x, t) \\ \frac{v p_j \cdot \sigma^j}{E_v + v p_4} \varphi(x, t) \end{pmatrix} \\ &= N \begin{pmatrix} \varphi_0 \\ \frac{v p_j \cdot \sigma^j}{E_v + v p_4} \varphi_0 \end{pmatrix} e^{\frac{i}{\hbar}(p \cdot x - E_v t)}. \end{aligned} \quad (32)$$

By calculating the normalization constant N for positive and negative energy solutions, we find

$$N = \sqrt{\frac{E_v + v p_4}{2E_v}}.$$

The two states that represents free moving electron are

$$\begin{aligned} u_1 &= \sqrt{\frac{E_v + v p_4}{2E_v}} \begin{pmatrix} 1 \\ 0 \\ \frac{v p_3}{E_v + v p_4} \\ \frac{v(p_1 + i p_2)}{E_v + v p_4} \end{pmatrix} \\ u_2 &= \sqrt{\frac{E_v + v p_4}{2E_v}} \begin{pmatrix} 0 \\ 1 \\ \frac{v(p_1 - i p_2)}{E_v + v p_4} \\ \frac{-v p_3}{E_v + v p_4} \end{pmatrix}. \end{aligned} \quad (33a)$$

If we choose the negative square root $E_v = -\sqrt{v^2 \mathbf{p}^2 + (v p_4)^2}$, similarly we obtain

$$\begin{aligned} u_3 &= \sqrt{\frac{E_v + v p_4}{2E_v}} \begin{pmatrix} \frac{v(p_1 - i p_2)}{E_v - v p_4} \\ \frac{-v p_3}{E_v - v p_4} \\ 0 \\ 1 \end{pmatrix}, \\ u_4 &= \sqrt{\frac{E_v + v p_4}{2E_v}} \begin{pmatrix} \frac{v p_3}{E_v - v p_4} \\ \frac{v(p_1 + i p_2)}{E_v - v p_4} \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \quad (33b)$$

We have obtained the same spinors of Dirac except that c is replaced with v . Although we started from classical electrodynamics, the analysis in our paper is still entirely relativistic by the formula $E_v = mv^2$ instead of $E_v = mc^2$. And we will show now that the modified Dirac equation, Eq. (25), leads to the same result of Bakhom concerning the “Zitterbewegung”. Furthermore, we reveal that the spin of the electron and its magnetic moment can be derived without using any kind of approximation.

Calculating the velocity from the modified Dirac Hamiltonian

In the Dirac relativistic equation for spin 1/2 particle, there is velocity operator $\hat{v} = c \boldsymbol{\alpha}$. It is believed that this operator is inadequate in two aspects. The first one is that its eigenvalues are $+c$ and $-c$, with c being the light speed in vacuum, The other is that it is not proportional to the linear momentum. To overcome these shortcomings, E.G. Bakhom obtained a Hamiltonian that is written as follows [6]

$$H = \pm v \mathbf{p}_r \cdot \boldsymbol{\beta}_r,$$

where β_r are matrices that satisfy these two conditions

$$\beta_r^2 = I, \text{ and } \beta_j \beta_k + \beta_k \beta_j = 0.$$

Unlike the Dirac result for the eigenvalue of the particle velocity, Bakhoum's result is in agreement with the experimental observation since

$$\dot{x} = [x, H] = \frac{\partial H}{\partial p_x} = v \beta_1.$$

From this equation it is clear that \dot{x} will be $+v$ or $-v$.

Another prominent attempt to eliminate the “*Zitterbewegung*” problem was by Recami *et al.* [12,13,14]. They showed that the “*Zitterbewegung*” is necessary for the quantum phenomenon of spin and gave it a physical (classical) meaning. In this paper, also without change to the standard theory, it is shown that by the modified Dirac Hamiltonian the velocity operator really retrieves the classical relation between velocity and momentum.

The problem of the “*Zitterbewegung*” in Dirac's Hamiltonian corresponds to the term $c\boldsymbol{\alpha}\cdot\mathbf{p}$. So, we will see how we get the expected value of the particle velocity from Eq. (25) without the problem of the “*Zitterbewegung*”. Multiplying Eq. (25) from the left with ψ^+ , we obtain

$$i\hbar \psi^+ \frac{\partial}{\partial t} \psi = -i\hbar v \psi^+ \alpha_j \nabla^j \psi + v p_4 \psi^+ \alpha_4 \psi. \quad (34)$$

Taking the hermitean conjugate of Eq. (25), we get

$$-i\hbar \frac{\partial}{\partial t} \psi^+ = i\hbar v \nabla_j \psi^+ \alpha_j + v p_4 \psi^+ \alpha_4. \quad (35)$$

Multiplying the last equation from the right with ψ , we have

$$-i\hbar \frac{\partial}{\partial t} \psi^+ \psi = i\hbar v \nabla_j \psi^+ \alpha_j \psi + v p_4 \psi^+ \alpha_4 \psi. \quad (36)$$

Subtracting Eq. (36) from Eq. (34), we get

$$\frac{\partial}{\partial t} (\psi^+ \psi) + \nabla_j (\psi^+ v \alpha_j \psi) = 0. \quad (37)$$

Comparing this equation with the familiar continuity equation

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{J} = 0 \quad (38)$$

we identify that

$$\mathbf{J} = v \psi^+ \boldsymbol{\alpha} \psi. \quad (39)$$

According to the known formula $\mathbf{J} = \rho \mathbf{v}$, we deduce that the expected value of the particle velocity obtained here from the modified Dirac equation equals $\pm v$.

The Dirac Hamiltonian “*Zitterbewegung*” results from interference between two positive and two negative energy components of the Dirac spinor. In resemblance to the solutions in the Dirac basis, the modified Dirac Hamiltonian “*Zitterbewegung*” results also from interference between two positive and two negative energy components of the modified Dirac equation solutions. It had been shown that the expected value of the particle velocity obtained here from the modified Dirac equation always equals the velocity that the particle moves with, as observed in laboratory a force-free electron can move at any velocity less than c . The difference between the interpretation of the “*Zitterbewegung*” from our modified Dirac Hamiltonian and from the Dirac Hamiltonian is now to be expected.

Interaction with the electromagnetic field

The most important result of the Dirac equation was presentation of a theoretical description of the electron spin and its magnetic moment that

did not appear directly from the Dirac equation but only after using approximations (Pauli, Foldy-Wouthysen). This is not the case when we derive the electron spin and its magnetic moment through the new Hamiltonian, Eq. (24).

As we know, to include the effects of the electromagnetic field the momentum p_μ is replaced in the following way

$$p_\mu \rightarrow p_\mu - \frac{e}{c} A_\mu. \quad (40)$$

We apply the same idea to the Hamiltonian in Eq. (24) which can be written as

$$H = v\alpha_\mu (p^\mu - \frac{e}{c} A^\mu); \quad \alpha_4 = \beta. \quad (41)$$

Following the method of Bakhoun in his paper [6], by squaring Eq. (41) and dividing it on mv^2 then we get this scalar equation

$$H = \frac{1}{m} [\alpha_\mu (p^\mu - \frac{e}{c} A^\mu)]^2. \quad (42)$$

In operator formalism, we get also the same result since the following eigenvalue equation holds

$$\hat{H}\psi = mv^2\psi. \quad (43)$$

If we define the operator \hat{G} as $(\hat{p}^\mu - \frac{e}{c} A^\mu)$, then Eq. (41) can be restated as $\hat{H} = v\alpha_\mu \hat{G}$ with mv^2 is an eigenvalue of \hat{H} . Because I do not know the operator \hat{v} so I do not know if it commutes with \hat{G} , which is surely necessary for the assignment $H^2 = v^2 G^2$. Further derivation bears out the behavior of \hat{v} that commutes with everything around. Hence by squaring Eq. (41) we get

$$H^2 = v^2 [\alpha_\mu (\hat{p}^\mu - \frac{e}{c} A^\mu)]^2. \quad (44)$$

From Eq. (41) the Dirac Hamiltonian for particle in electromagnetic field, (A_4 is set equal ϕ), is

$$\hat{H} = v(\alpha_j \hat{p}^j - \frac{e}{c} \alpha_j A^j) + v\alpha_4 p_4 - \frac{e}{c} v\phi.$$

Further, the motion equation for the \hat{x} operator is given through

$$\frac{d\hat{x}_j}{dt} = \frac{1}{i\hbar} [\hat{H}, \hat{x}_j]. \quad (45)$$

The commutator in Eq. (45) is

$$[\hat{H}, \hat{x}_j] = v[\hat{\alpha}_j \hat{p}^j, \hat{x}_j] - \frac{e}{c} v[\hat{\alpha}_j A^j, \hat{x}_j] + v p_4 [\hat{\alpha}_4, \hat{x}_j] - \frac{e}{c} v[\phi, \hat{x}_j].$$

Since \hat{x}_j operators commute with both α_4 and A_4 ; further $[\alpha_j A^j, x_j] = 0$, and because \hat{x}_j commute with both α_j and A_j , then we get

$$\frac{d\hat{x}_j}{dt} = \frac{1}{i\hbar} v[\alpha_j \hat{p}^j, \hat{x}_j] = \frac{1}{i\hbar} v\alpha_j [p^j, \hat{x}_j] = v\alpha_j.$$

That means, the velocity operator of Dirac particle is

$$\hat{v} = v\hat{\alpha}. \quad (46)$$

The operator $\hat{v} = v\hat{\alpha}$ is more convenient than $\hat{v} = c\hat{\alpha}$, because Eq. (46) now depends on actual velocity v and the velocity v in Eq. (46) is the speed of free electron, as well as in Eqs. (24,25) should be defined as ‘‘average electron speed’’, [16]. We use Eq.(46) to find that the operator \hat{H} commute with \hat{v}

$$[\hat{H}, \hat{v}] = [v(\alpha_j \hat{p}^j - \frac{e}{c} \alpha_j A^j) + v\alpha_4 p_4 - \frac{e}{c} v\phi, \hat{v}] = 0.$$

Therefore, we can now compute the following eigenvalue equation

$$\hat{H}^2\psi = \hat{H}(\hat{H}\psi) = m^2 v^2 \hat{H}\psi.$$

From another side, by operating with Eq. (44) on ψ we obtain

$$\hat{H}^2 \psi = v^2 [\alpha_\mu (\hat{p}^\mu - \frac{e}{c} A^\mu)]^2 \psi.$$

Comparing the last two equations, then we have

$$m v^2 \hat{H} \psi = v^2 [\alpha_\mu (\hat{p}^\mu - \frac{e}{c} A^\mu)]^2 \psi. \quad (47)$$

By dividing Eq. (47) on $m v^2$ then we get Eq. (42)

$$\hat{H} = \frac{1}{m} [\alpha_\mu (\hat{p}^\mu - \frac{e}{c} A^\mu)]^2.$$

In the reference [2] the following equation holds

$$[\alpha_\mu (\hat{p}^\mu - \frac{e}{c} A^\mu)]^2 = (\hat{p}_\mu - \frac{e}{c} A_\mu)^2 - \frac{e \hbar}{c} \sigma_4 \cdot \mathbf{B}, \quad (48)$$

where the matrices σ_4 in Eq. (48) are the 4×4 Pauli matrices which have the following representation

$$\sigma_4 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

Substituting Eq. (48) in Eq. (47) we finally obtain

$$\hat{H} = \frac{1}{m} (\hat{p}_\mu - \frac{e}{c} A_\mu)^2 - \frac{e \hbar}{m c} \sigma_4 \cdot \mathbf{B}. \quad (49)$$

Deriving relativistic Pauli equation without approximation methods

If we consider only the effect of a magnetic field \mathbf{B} then the momentum is replaced as $\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c} \mathbf{A}$, and Eq. (49) becomes

$$\hat{H} = \frac{1}{m} \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A} \right)^2 - \frac{e \hbar}{m c} \sigma_4 \cdot \mathbf{B}. \quad (50)$$

The second term on the r.h.s. of Eq. (50) represents the interaction of the electron spin magnetic moment with the magnetic field. An important characteristic of Eq. (50) is that we did not use any kind of approximation to reach it. Therefore, we got the relativistic mass m not the rest mass m_0 , also the Pauli matrices here σ_4 are 4×4 matrices. That is why Eq. (50) can be regarded as a relativistic Pauli equation.

The usual Pauli equation can be obtained from Eq. (50) by canceling terms containing v/c , since for $m \approx m_0$ we have $H = m_0 v^2$, so we get the Pauli equation

$$\frac{1}{2} m_0 v^2 = \frac{1}{2 m_0} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + \frac{e \hbar}{2 m_0 c} \sigma \cdot \mathbf{B}. \quad (51)$$

Without the presence of the magnetic field Eq. (51) is reduced to

$$\frac{\mathbf{p}^2}{2 m_0} = \frac{1}{2} m_0 v^2.$$

That means the electron has a spin magnetic moment $\boldsymbol{\mu} = -\frac{e \hbar}{2 m_0 c} \boldsymbol{\sigma}$, and the magnetic moment interacts with an external magnetic field, so the corresponding contribution to the energy is $-\boldsymbol{\mu} \cdot \mathbf{B}$. We recognize here the classical Pauli equation for the theory of spin with the correct gyromagnetic factor $g = 2$ for the electron.

Discussion

There exists an inconsistency between Einstein's special relativity and the De Broglie wave mechanics and it has never been resolved from the viewpoint of relativistic physics for a long time [15]. A more suitable method to deal with

this contradiction is to develop the applicability of the classical physics laws to all particle velocities i.e. to expand the appropriateness of these laws to deal with the relativistic domain. Following this approach, a modified Dirac equation can be derived using classical description as it is shown in this paper. Bakhoun merely showed how the modern physics as we know it can be understood on the basis of the equation $H = mv^2$. In particular, Einstein's equation $H = mc^2$ becomes a special case of the broader equation. The work of Hamdan *et al* [8] carries Bakhoun's work a step further since recently we derived the formula without using the special relativity theory too, but starting from the Lorentz force law and the relativity principle. In this paper, we have derived the modified Dirac equation to obtain the same result of Bakhoun concerning "Zitterbewegung". Further, we have gained additional advantage, where the modified Dirac equation, Eq. (25), will directly lead to the relativistic Pauli equation without using any kind of approximation methods. Unlike the Dirac equation and its predications, our modified Dirac equation can be derived using classical description and its results remove the conceptual difficulty with the problem of the "Zitterbewegung" and approximation methods.

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