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APPLICATION OF COLLOCATION APPROXIMATIONS FOR THE SOLUTION OF FOURTH-ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract: This paper concerns the application of collocation approximation for the solution of fourth-order nonlinear boundary value problems by Chebyshev polynomial approximation. A new basis function for a collocation solution called Chebyshev basis via standard and Perturbed methods was used. Methods discussed in the paper cannot handle nonlinear directly; hence the nonlinear cases treated by the Newton's linearization scheme of order 4. Numerical examples were given to illustrate the effectiveness of the methods discussed in this paper. Mathematical software 'MATLAB' was used to solve the algebraic system of equations obtained in the illustrative examples.

Keywords: Chebyshev polynomial, collocation solution, standard and perturbed methods, Newton's linearization, nonlinear. Matlab.

Introduction

Nonlinear differential equations are used in modeling many real life problems in science and engineering. Fourth order boundary value problems occur in a number of areas of applied mathematics, among which are fluid mechanics elasticity and quantum mechanics.

The fourth-order boundary value problems considered in this paper are of the form:

$$y^{iv}(x) = f\begin{pmatrix} x\ ,\ y(x),\ y'(x), \\ y''(x),\ y'''(x) \end{pmatrix} y'(x) \tag{1.1}$$

which are valid in some interval a < x < b, together with conditions imposed on the dependent variable at the two end points x = a and x = b and f is, in general, a nonlinear function.

Recently, several authors have investigated solving fourth-order boundary value problem by some numerical techniques which include the Cubic Spline Method, Ritz method, finite difference method, also multiderivative methods and Finite element methods [1,2,3,4]. In this, paper, we have used the standard and Perturbed collocation methods on the linearized form of equation (1.1).

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Newton's Linearization Techniques

The Newton's scheme from the Taylor's series expansion of order 4 is given by

$$\begin{aligned} & G + \Delta y \frac{\partial G}{\partial y} + \Delta y' \frac{\partial G}{\partial y'} + \Delta y'' \frac{\partial G}{\partial y''} + \\ & \Delta y''' \frac{\partial G}{\partial y'''} + \Delta y^{iv} \frac{\partial G}{\partial y^{iv}} = 0 \end{aligned} \tag{1.2}$$

is used throughout in this paper. Thus, from equation (1.1), we obtain the following:

$$\left.\begin{aligned} \frac{\partial G}{\partial y} &= -f_{y_k}\ y'_k \\ \frac{\partial G}{\partial y'_k} &= -\left(fy'_k\right) y'_k - f_k \\ \frac{\partial G}{\partial y''_k} &= -fy''_k\ y'_k \\ \frac{\partial G}{\partial y'''_k} &= -fy'''_k\ y'_k \\ \frac{\partial G}{\partial y^{iv}_k} &= 1 \end{aligned}\right\} \qquad (1.3$$

Substituting equation (1.3) into equation (1.2), we obtain

$$G_k + \Delta y_k \left(- fy_k \ y_k'\right) + \Delta y_k' \left(- fy_k' \ y_k' - f_k\right) + \Delta y_k'' \left(- fy_k'' \ y_k'\right) + \Delta y_k''' \left(- fy_k''' \ y_k'\right) + \Delta y_k^{iv} (1) = 0 \quad (1.4)$$

Here, we define

$$\Delta y_k^{(j)} = \Delta y_{k+1}^{(j)} - y_k^{(j)} \ , \ 1 \le j \le 4 \quad (1.5)$$

and j is the order of the derivative. Thus, the Newton's linearization scheme leads to the following iterations

$$y_{k+1}^{iv} - \left(fy_k''' \ y_k'\right) y_{k+1}''' - \left(fy_k'' \ y_k'\right) y_{k+1}'' - \left(f + f_k \ y_k'\right) y_{k+1}' - \left(f_k \ y_k'\right) y_{k+1}'' = y_k^{iv} - fy_k''' \ y_k' \ y_k''' - fy_k'' \ y_k' \ y_k'' - \left(f_k + fy_k' \ y_k'\right) y_k' - fy_k' \ y_k' \ y_k - G_k \quad (1.6)$$

Hence, equation (1.6) is our linearized form of equation (1.1).

Numerical Solution Techniques

Method 1: Standard Collocation Method

In order to apply this method to solve our linearized equation (1.6), we assume approximate solution of the form:

$$y_{N,k+1}(x) = \sum_{i=0}^{N} a_i \ T_i(x) \quad (1.7)$$

where a_i (i = 0(1) N) are constant to be determined and $T_i(x)$ are the set of Chebyshev polynomials defined by

$$T_i(x) = \cos\{i \cos^{-1} x\}; \ -1 \le x \le 1 \quad (1.8)$$

Thus, substituting equation (1.7) into equation (1.6), we obtain

$$\sum_{i=0}^{N} a_i \ T_i^{iv}(x) - \left(fy_k''' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i'''(x) - \left(fy_k''' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i''(x) - \left(f_k + fy_k' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i'''(x) - fy_k \ y_k' \sum_{i=0}^{N} a_i \ T_i(x) = y_k^{iv} - fy_k''' \ y_k' \ y_k''' - fy_k'' \ y_k' \ y'' \ \left(f_k + fy_k' \ y_k'\right) y_k' - fy_k' \ y_k - G_k \quad (1.9)$$

Hence, collocating equation (1.9) at point $x = x_i$, we obtain

$$\sum_{i=0}^{N} a_i \ T_i^{iv}(x_j) - \left(fy_k''' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i''(x_j) - \left(fy_k''' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i''''(x_j) - \left(f_k + fy_k' \ y_k'\right) \sum_{i=0}^{N} a_i \ T_i'''(x_j) - fy_k \ y_k' \sum_{i=0}^{N} a_i \ T_i(x) = y_k^{iv} - fy_k''' \ y_k' \ y_k''' - fy_k'' \ y_k' \ y'' \ \left(f_k + fy_k' \ y_k'\right) y_k' - fy_k' \ y_k - G_k \quad (1.10)$$

where for some obvious practical reasons, we choose the collocation points

$x = x_j$ to be

$$x_j = a + \frac{(b-a)}{N} \ ; \ j = 1, 2, \ldots, N-1 \quad (1.11)$$

Thus, we have N collocation equations in (N + 3) unknowns. Four extra equations will be obtained from the given boundary conditions of the problems. Altogether, we now have (N + 3) collocation equations which give the unique values of the (N + 3) constant $a_0, a_1, a_2, a_3, \ldots, a_N$. These (N + 3) algebraic linear systems are then solved to obtain the (N + 1) unknown constants a_i (i = 0(1) N) which are substituted into our approximation equation (1.7) together with the values of the Chebyshev polynomials.

Method 2: Perturbed Collocation Method

The perturbed collocation method is an attempt to improve the accuracy and efficiency of the standard collocation method. In order to discuss this method, we again re-visit our linearized equation (1.6) and assume an approximation of equation (1.7). Thus, equation (1.7) is substituted into a slightly perturbed equation (1.6), and we obtain

$$\sum_{i=0}^{N} a_i\ T_i^{iv}(x) - (fy_k'''\ y_k')\ \sum_{i=0}^{N} a_i\ T_i'''(x) -$$
$$(fy_k'''\ y_k')\sum_{i=0}^{N} a_i\ T_i''(x) - (f_k + fy_k'\ y_k')$$
$$\sum_{i=0}^{N} a_i\ T_i'(x) - fy_k\ \ y_k'\ \sum_{i=0}^{N} a_i\ T_i(x) = \qquad (1.12)$$
$$y_k^{iv} - fy_k'''\ y_k'\ y_k''' - fy_k''\ y_k'\ y'''$$
$$(f_k + fy_k'\ \ y_k')\,y_k' - fy_k'\ \ y_k - G_k +$$
$$\tau_1\ P_N(x) + \tau_2\ P_{N-1}(x) + \tau_3\ P_{N-3}(x) +$$
$$\tau_4\ P_{N-4}(x); \quad a \le x \le b$$

where $P_N(x)$ is an orthogonal polynomial of degree N (Chebyshev polynomial for a single polynomial approximation over [a , b] or Legendre polynomial for piece wise polynomial approximation over [a, b] and τ_i (i = 1(1) 4) are free parameters to be determined. Thus, collocating equation (1.12) at points x = x_i , yields

$$\sum_{i=0}^{N} a_i\ T_i^{iv}(x_i) - (fy_k'''\ y_k')\ \sum_{i=0}^{N} a_i\ T_i'''(x) -$$
$$(fy_k'''\ y_k')\sum_{i=0}^{N} a_i\ T_i''(x_i) - (f_k + fy_k'\ y_k')$$
$$\sum_{i=0}^{N} a_i\ T_i'''(x_i) - fy_k\ \ y_k'\ \sum_{i=0}^{N} a_i\ T_i''(x_i) = \qquad (1.13)$$
$$y_k^{iv} - fy_k'''\ y_k'\ y_k''' - fy_k''\ y_k'\ y'''$$
$$(f_k + fy_k'\ \ y_k')\,y_k' - fy_k'\ \ y_k - G_k +$$
$$\tau_1\ P_N(x_i) + \tau_2\ P_{N-1}(x_i)$$
$$+ \tau_3\ P_{N-3}(x_i) + \tau_4\ P_{N-4}(x_i); \ a \le x_i \le b$$

where, for some obvious practical reasons, we choose the collocation point x = x_i to be,

$$x_i = a + \frac{(b-a)i}{N+4}\ ; \ i = 1\ ,\ 2, \ldots ,\ N + 3 \quad (1.14)$$

Hence, equation (1.13) gives (N + 3) collocation equations in (N + 7) unknowns. Thus four extra equations are obtained from the boundary conditions imposed. Altogether we have (N + 7) algebraic equations which give the unique values of the (N + 7) constants a_i (i = 0(1) N) and τ_k (k = 1(1) 4) to obtain a single polynomial approximation.

Numerical Examples and Discussion of Results

In this section, we shall consider the following examples and also define our error as absolute value (y(x) - $y_{N,k+1}(x)$)

Example 1

We solve the fourth order nonlinear boundary value problem given as:

$$y^{iv}(x) = (y'(x))^2 - y(x)\,y'''(x) -$$
$$4x^2 + e^x\left(1 + x^2 - 4x\right) \ 0 \le x \le 1$$

together with the boundary conditions

y(0) = 1
y(1) = 1 + e
y′(0) = 1
and
y′(1) = 2 + e

The exact solution y(x) is given as $x^2 + e^x$. Thus, using the Newton’s linearization scheme to linearize the given problem, we obtain

$$y_{k+1}^{iv} + y_k\ y_{k+1}''' - 2y_k'\ y_{k+1}' +$$
$$y_k'''\ y_{k+1} = (y_k')^2 - y_k\ y_k''' - 4x^2$$
$$+ e^x\left(1 + x^2 - 4x\right) \quad 0 \le x \le 1$$

together with the boundary conditions

$y_{k+1}(0) = 1$
$y_{k+1}(1) = 1 + e$
$y'_{k+1}(0) = 1$
and
$y'_{k+1}(1) = 2 + e$

We have used an initial guess $y_0(x) = 2x + e^x$.

Example 2

We solve the fourth order nonlinear boundary value problem given as:

$$y^{iv}(x) - e^{x}\ y'''(x) - y(x)\, y'''(x) = 2xe^{x} - e^{x}; \qquad 0 \le x \le 1$$

or

$$y^{iv}(x) - \left(e^{x}\ y'''(x)\right) y'''(x) = (2x-1)e^{x}; \qquad 0 \le x \le 1$$

together with the boundary conditions

$y(0) = 1$
$y(1) = 2 - e$
$y'(0) = 1$
and
$y'(1) = 2 - e$

The exact solution y(x) is given as $2x - e^{x}$. Thus, using the Newton's linearization scheme to linearize the given problem, we obtain

$$y^{iv}_{k+1}(x) - e^{x}\ y''_{k+1}(x) - y_{k}(x)\, y_{k+1}(x) = y_{k}(x)\, y''_{k} - (2x+1)e^{x}; \qquad 0 \le x \le 1$$

together with the boundary conditions

$y_{k+1}(0) = 1$
$y_{k+1}(1) = 2 - e$
$y'_{k+1}(0) = 1$
and
$y'(1) = 2 - e.$

Here, we have used an initial guess $y_0(x) = 2x + e^{x}$.

Tables of Results for the examples considered

Table 1. Example 1 for Case N = 4 at seventh iterations.

x	Error of Standard Collocation Method	Error of Perturbed Collocation Method
0	0	0
0.1	8.5036E-4	1.2054E-4
0.2	6.2483E-3	4.1815E-3
0.3	4.3211E-3	1.0496E-3
0.4	3-2112E-3	1.0234E-3
0.5	3.1255E-3	1.0127E-3
0.6	2.1249E-3	1.0125E-3
0.7	2.1400E-3	1.0123E-3
0.8	5.0213E-2	1.0122E-3
0.9	5.3213E-2	6.2357E-4
1.0	5.6543E-5	5.4523E-7

Table 2. Example 1 for Case N = 5 at seventh iterations.

X	Error of Standard Collocation Method	Error of Perturbed Collocation Method
0	0	0
0.1	3.1025E-3	1.1749E-4
0.2	9.0439E-3	2.8650E-4
0.3	7.7109E-3	8.1444E-4
0.4	3.0582E-3	1.5790E-3
0.5	2.6251E-2	1.2319E-3
0.6	2.4719E-2	2.5220E-3
0.7	1.9326E-2	4.7805E-3
0.8	1.1450E-2	9.3729E-4
0.9	4.2569E-3	3.4968E-5
1.0	6.7542E-7	2.5437E-9

Table 3. Example 1 for Case N = 6 at seventh iterations.

X	Standard Collocation Method	Perturbed Collocation Method
0	0	0
0.1	8.4499E-4	3.6890E-4
0.2	6.3757E-3	6.3756E-4
0.3	9.1852E-5	1.1176E-5
0.4	1.0421E-3	1.5415E-4
0.5	1.2190E-3	2.0973E-4
0.6	1.4984E-2	9.6741E-4
0.7	7.1436E-3	5.8214E-4
0.8	4.2899E-3	2.9659E-4
0.9	2.6885E-4	9.5094E-5
1.0	5.4800E-7	1.0089E-9

Table 4. Example 2 for Case N = 4 at seventh iterations.

X	Error of Standard Collocation Method	Error of Perturbed Collocation Method
0	0	0
0.1	4.4449E-4	8.5036E-5
0.2	1.3731E-3	6.2483E-4
0.3	2.3088E-3	1.0812E-3
0.4	2.9394E-3	4-1997E-4
0.5	3.1320E-3	1.2549E-4
0.6	1.5440E-3	1.4358E-4
0.7	1.0251E-3	1.1057E-4
0.8	3.9217E-3	6.4573E-4
0.9	2.1098E-4	2.0558E-4
1.0	2.4950E-5	5.4532E-8

Table 5. Example 2 for Case N = 5 at seventh iterations.

X	Standard Collocation Method	Perturbed Collocation Method
0	0	0
0.1	3.1025E-3	1.1749E-6
0.2	9.0437E-3	2.8650E-5
0.3	7.7107E-3	7.6511E-5
0.4	3.0582E-3	9.2311E-5
0.5	2.6251E-2	1.2319E-4
0.6	2.4719E-2	3.5220E-4
0.7	1.9326E-2	4.7805E-4
0.8	1.1450E-2	9.3729E-5
0.9	4.2569E-3	4.4968E-6
1.0	1.6743E-8	1.0021E-9

Table 6. Example 2 for Case N = 6 at seventh iterations.

X	Standard Collocation Method	Perturbed Collocation Method
	0	0
0.1	8.4499E-4	3.6870E-7
0.2	6.3757E-3	6.3756E-7
0.3	9.1852E-5	1.1176E-6
0.4	1.0421E-3	1.5415E-5
0.5	1.2190E-3	2.0973E-5
0.6	1.4984E-2	3.6741E-5
0.7	7.1436E-3	5.8214E-5
0.8	4.2899E-3	6.9659E-5
0.9	2.6885E-4	9.5094E-6
1.0	5.4811E-7	1.0051E-9

Conclusion

Two numerical collocation methods have been described and applied to nonlinear fourth order Boundary Value Problems. The two collocation methods are compared favourably with the exact solutions. We also observed that as N increases, Perturbed collocation method converges faster than the Standard Collocation method. In terms of computations, Perturbed collocation method involved more work because more unknowns to be determined are involved. Thus, Perturbed collocation method is highly recommended for the type of problems considered in this work as the extra work involved is being compensated for in terms of accuracy achieved. Comparison of the two methods is also illustrated graphically.

References

1. **Twizell, E.H. and Tirmizi, S.I.A**. 1984. *Multi-derivative methods for linear fourth order boundary value problems.* Technical Report. Dept. of Maths. and Stats. Brunel University, TR 06.
2. **Sakai, M. and Usmani, R.** 1983. *Spline solutions for nonlinear fourth – order two-point boundary value problems.* Publ. RIMS, Kyoto University. 19:135 - 144.
3. **Ortiz, E.L.** 1968. The Tau method; on numerical analysis *SIAM J.* 6:480-492.
4. **Usmani, R.A. and Taylor, P.J.** 1983. Finite difference methods for solving $[P(x)\, y''']' + q(x)\, y = r(x)$. *Intern. J. Computer Maths.* 14:227-293.
5. **Taiwo, O.A. and Evans, D.J**. 1997. Collocation approximation for fourth-order boundary value problems. Intern. *J. Computer Math.* 63:57-66.