

FERMIONS IN KERR-NEWMAN-KASUYA SPACE-TIME

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Abstract: The aim of this paper is to build up the U(1)- gauge theory for fermions in the curve space-time such as Kerr-Newman-Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

Keywords: Gauge theory, black hole space-time, horizon, magnetic monopole charge, null complex tetrads

Introduction

Carmeli and Carmeli derived the Klein-Gordan, Weyl, and Dirac-type equations on $R \times S^3$ topology by simply going from the momentum to the angular momentum representation [1,2,3]. Sen [4] obtained the most general Lagrangians for the Dirac, Weyl, and Majorana fermions. Sen's work offers an excellent description of fermions in the space-time $R \times S^3$. Dariescu *et al.* [4] developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman black hole space-time. Dariescu and Dariescu [5] also developed the tetradic Lorentz-gauge invariant formulation of the $SU(2) \times U(1)$ theory in $R \times S^3$ space-time. To develop a U(1)-gauge theory for massive fermionic fields on curved space-time such as Kerr-Newman black hole space-time, they used the Dirac-type equation and the U(1)-gauge invariant Lagrangian. In this paper, the U(1)-gauge theory for massive fermionic fields minimally coupled to a Kerr-Newman-Kasuya space-time is studied. The Kerr-Newman-Kasuya

space time is the Kerr-Newman space-time involved with extra magnetic monopole charge. This monopole hypothesis was propounded by Dirac relativity long ago.

The Kerr-Newman-Kasuya space-time

The Kerr-Newman-Kasuya space-time is described by the metric

$$ds^2 = (r^2 + h^2 \cos^2 \theta) \left(\frac{dr^2}{r^2 - 2mr + e^2 + h^2 + l^2} + d\theta^2 \right) + \sin^2 \theta \left\{ r^2 + h^2 + \frac{h^2 \sin^2 \theta (2mr - e^2 - l^2)}{r^2 + h^2 \cos^2 \theta} \right\} d\varphi^2 \quad (1)$$

$$- \left(1 - \frac{2mr - e^2 - l^2}{r^2 + h^2 \cos^2 \theta} \right) dt^2 - \frac{2h \sin^2 \theta (2mr - e^2 - l^2)}{r^2 + h^2 \cos^2 \theta} dt d\varphi$$

where m , h , e and l are the mass, angular momentum per unit mass, electric charge and magnetic monopole charge parameters respectively. This is a solution to the Einstein-Maxwell equations with electromagnetic vector potential

$$A_\mu dx^\mu = \frac{er (dt - h \sin^2 \varphi d\varphi)}{r^2 + h^2 \cos^2 \theta} \quad (2)$$

The space-time given by (1) encompasses all the black hole space-times, which are asymptotically flat. Specially, the metric (1) includes:

- (i) Kerr-Newmann black hole space-time when $l=0$.
- (ii) Kerr black hole space-time for $l=e=0$.
- (iii) Reissner-Nordstrom black hole space-time if $l=h=0$.
- (iv) Schwarzschild black hole space-time when $l=e=h=0$.

This metric can be transformed to Boyer coordinates under the proper coordinate transformation such as

$$\{x^\mu\} = \left\{ p = h \cos \theta ; \sigma = -\frac{\phi}{h} ; q = r ; \tau = t - h\phi \right\} \quad (3)$$

with the suitable adjustment of the parameter

$$X(p) = h^2 - p^2 ; Y(q) = q^2 - 2mq + e^2 + h^2 + l^2 \quad (4)$$

Therefore, the metric (1) can be written as

$$ds^2 = \frac{p^2 + q^2}{X} dp^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma)^2 + \quad (5)$$

$$\frac{p^2 + q^2}{Y} dq^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma)^2$$

This metric represents the Kerr-Newman-Kasuya space-time in Boyer coordinates, which has been studied in detail by Plebanski [7]. After a suitable choice of the null complex tetrads $\{\omega^a\}$ which consists of two complex conjugate null vectors m, \bar{m} and two real null vectors

$$k_1, k_2 ; \{\omega^a\} = \{m, \bar{m}, k_1, k_2\} :$$

$$\begin{aligned} \omega^1 = m &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{p^2 + q^2}{X}} dp + i \sqrt{\frac{X}{p^2 + q^2}} (d\tau + q^2 d\sigma) \right] \\ \omega^2 = \bar{m} &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{p^2 + q^2}{X}} dp - i \sqrt{\frac{X}{p^2 + q^2}} (d\tau + q^2 d\sigma) \right] \\ \omega^3 = k_1 &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{Y}{p^2 + q^2}} (d\tau - p^2 d\sigma) - \sqrt{\frac{p^2 + q^2}{Y}} dq \right] \\ \omega^4 = k_2 &= \frac{1}{\sqrt{2}} \left[\sqrt{\frac{Y}{p^2 + q^2}} (d\tau - p^2 d\sigma) + \sqrt{\frac{p^2 + q^2}{Y}} dq \right] \end{aligned} \quad (6)$$

the metric (1) becomes in the simple form

$$ds^2 = 2(\omega^1 \omega^2 - \omega^3 \omega^4) = g_{ab} \omega^a \omega^b \quad (7)$$

with

$$(g_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (8)$$

Field equations

For the massive fermionic complex fields ψ the U(1)-gauge invariant Lagrangian is given by

$$L = \bar{\psi} \gamma^\mu D_\mu \psi + M \bar{\psi} \psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (9)$$

where γ^μ is the generalized Dirac gamma metrics, the U(1)-gauge field-strength tensor is defined as

$$F^{\mu\nu} = g^{\mu\alpha} \partial_\alpha A^\nu - g^{\nu\alpha} \partial_\alpha A^\mu - (g^{\mu\alpha} \partial_\alpha g^{\nu\beta} - g^{\nu\alpha} \partial_\alpha g^{\mu\beta}) g_{\beta\sigma} A^\sigma \quad (10)$$

and the gauge-covariant derivative is defined as

$$D_\mu \psi = \nabla_\mu \psi + ig A_\mu \psi, \text{ and its h.c.} \quad (11)$$

here $\nabla_\mu \psi$ be the Levi-Civita covariant derivative and g is the gauge coupling constant. Under these assumptions, the Dirac-type equation is obtained in the covariant expression

$$\gamma^\mu (\partial_\mu + ig A_\mu) \psi - \frac{1}{4} \Gamma_{\alpha\beta\mu} \gamma^\mu \gamma^\alpha \gamma^\beta \psi + M \psi = 0 \quad (12)$$

The Dirac-type equation governs the particle in curved space-time, and the Maxwell equations with sources can be expressed in the standard form

$$\gamma^\mu (\partial_\mu + ig A_\mu) \psi - \frac{1}{4} \Gamma_{\alpha\beta\mu} \gamma^\mu \gamma^\alpha \gamma^\beta \psi + M \psi = 0 \quad (13)$$

which will be generalized for the case of a null tetradic base $\{e_a\}, a=1,4$. To build up a U(1)-gauge theory of a massive fermionic complex

field in the curved space-time described by the metric (7) we use the U(1)-gauge invariant Lagrangian. The general expression for the covariant derivative (11) becomes,

$$D_a \psi = \nabla_a \psi + ig A_a \psi, \text{ and its h.c.} \quad (14)$$

and the Lagrangian (9) as

$$L = \bar{\psi} \gamma^a D_a \psi + M \bar{\psi} \psi + \frac{1}{4} F_{ab} F^{ab} \quad (15)$$

The electromagnetic tensor F^{ab} can be expressed in the base of coordinate (p, q, σ, τ)

$$F^{ab} = \omega_\mu^a \omega_\nu^b F^{\mu\nu} \quad (16)$$

Therefore, the essential components $F^{\mu\nu}$ are given below

$$F^{12} = \frac{X}{p^2+q^2} A_p^2 - \frac{p^2+q^2}{q^4 X - p^4 Y} A_\sigma^1 - \frac{p^2+q^2}{q^2 X + p^2 Y} A_\tau^1 - \left[\frac{X(q^4 X - p^4 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^4 X - p^4 Y} \right)_p + \frac{X(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X + p^2 Y} \right)_p \right] A^2 \quad (17.1)$$

$$F^{13} = \frac{X}{p^2+q^2} A_p^3 - \frac{Y}{p^2+q^2} A_q^1 - \frac{X}{Y} \left(\frac{Y}{p^2+q^2} \right)_p A^3 + \frac{Y}{X} \left(\frac{X}{p^2+q^2} \right)_q A^1 \quad (17.2)$$

$$F^{14} = \frac{X}{p^2+q^2} A_p^4 - \frac{p^2+q^2}{q^2 X + p^2 Y} A_\sigma^1 - \frac{p^2+q^2}{X-Y} A_\tau^1 - \left[\frac{X(q^4 X - p^4 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X - p^2 Y} \right)_p + \frac{X(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_p \right] A^2 - \left[\frac{X(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_p + \frac{X(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X + p^2 Y} \right)_p \right] A^4 \quad (17.3)$$

$$F^{23} = -\frac{Y}{p^2+q^2} A_q^2 + \frac{p^2+q^2}{q^4 X - p^4 Y} A_\sigma^3 + \frac{p^2+q^2}{q^2 X + p^2 Y} A_\tau^3 + \left[\frac{Y(q^4 X - p^4 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^4 X - p^4 Y} \right)_q + \frac{Y(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X + p^2 Y} \right)_q \right] A^2 + \left[\frac{Y(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X + p^2 Y} \right)_q + \frac{Y(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^4 X - p^4 Y} \right)_q \right] A^4 \quad (17.4)$$

$$F^{24} = \frac{p^2+q^2}{q^4 X - p^4 Y} A_\sigma^4 + \frac{p^2+q^2}{q^2 X + p^2 Y} A_\tau^4 - \frac{p^2+q^2}{q^2 X + p^2 Y} A_\sigma^2 - \frac{p^2+q^2}{X-Y} A_\tau^2 \quad (17.5)$$

$$F^{34} = \frac{Y}{p^2+q^2} A_q^4 - \frac{p^2+q^2}{q^2 X + p^2 Y} A_\sigma^3 - \frac{p^2+q^2}{X-Y} A_\tau^3 - \left[\frac{Y(q^4 X - p^4 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X - p^2 Y} \right)_q + \frac{Y(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_q \right] A^2 - \left[\frac{Y(X-Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{X-Y} \right)_q + \frac{Y(q^2 X + p^2 Y)}{(p^2+q^2)^2} \left(\frac{p^2+q^2}{q^2 X + p^2 Y} \right)_q \right] A^4 \quad (17.6)$$

These components of $F^{\mu\nu}$ allow us to put the Maxwell equations (13) in the expression

$$e_a F^{ba} = J^b \quad (18)$$

Finally the Dirac-type equation is derived for the spinorial massive complex field ψ coupled to the Kerr-Newman-Kasuya space-time. Using the U(1)-gauge invariant Lagrangian (15) the Dirac-type equation is obtained in the general form

$$\gamma^a (\partial_a + ig A_a) \psi - \frac{1}{4} \Gamma_{bca} \gamma^a \gamma^b \gamma^c \psi + M \psi = 0 \quad (19)$$

Hence, by working out the above Dirac-type equation for the metric (5) can be expressed in the form

$$\gamma^a (\partial_a + ig A_a) \psi + M \psi - \frac{1}{4\sqrt{2(p^2+q^2)^3}} \times \left[\left\{ \frac{(p^2+q^2) \left(\frac{\partial X}{\partial p} \right) - 2pX}{\sqrt{X}} (\gamma^1 + \gamma^2) + \frac{(p^2+q^2) \left(\frac{\partial Y}{\partial q} \right) - 2qY}{\sqrt{Y}} (\gamma^3 - \gamma^4) \right\} \right] \psi = 0 \quad (20)$$

$$[-8i[p\sqrt{Y}\gamma^1\gamma^2(\gamma^3+\gamma^4) - q\sqrt{X}(\gamma^1-\gamma^2)\gamma^3\gamma^4]]$$

Using (3) and (4) into the equation (20) we obtain the Dirac-type equation for the metric (1) in the following form

$$\gamma^a (\partial_a + ig A_a) \psi + M \psi + \left[\frac{\{r^2 + h^2\} h \cos \theta (\gamma^1 + \gamma^2)}{2h \sin \theta \sqrt{2\{r^2 + h^2 \cos^2 \theta\}^3}} \right] \psi - \left[\frac{\{h^2 \cos^2 \theta (r-m) + r(rm - e^2 - h^2 - l^2)\} (\gamma^3 - \gamma^4)}{2\sqrt{2\{r^2 + h^2 \cos^2 \theta\}^3} (r^2 - 2mr + e^2 + h^2 + l^2)} \right] \psi + \left[\frac{2i\{h \cos \theta (\sqrt{r^2 - 2mr + e^2 + h^2 + l^2})\} (\gamma^3 + \gamma^4) \gamma^1 \gamma^2}{\sqrt{2\{r^2 + h^2 \cos^2 \theta\}^3}} \right] \psi - \left[\frac{2irh \sin \theta (\gamma^1 - \gamma^2) \gamma^3 \gamma^4}{\sqrt{2\{r^2 + h^2 \cos^2 \theta\}^3}} \right] \psi = 0 \quad (21)$$

The result obtained in this paper corresponds to the result obtained in the case of the Kerr Newman black hole space-time when $l = 0$. Under this observation, we like to claim that this study encompasses the known result of Dariescu *et al.* [4] in the context of Kerr-Newman black hole. So, it is interesting to note that the

U(1)-gauge theory for fermions not only exists in the Kerr-Newman black hole space-time, but also in the Kerr-Newman Kasuya space-time. The Kerr-Newman-Kasuya space-time is not a black hole space-time but it has the common feature with the black hole space-time that it has horizon.

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