SOME APPLICATIONS OF A LOVELOCK'S THEOREM

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Abstract: Here we employ a Lovelock´s result for a) to obtain the general structure of the second fundamental form of intrinsically rigid spacetimes of class one, b) to show a known identity between the Riemann tensor and its double dual, and c) to construct a Lanczos potential for the Gödel metric.

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Introduction

We shall use the notation and quantities of [1]. Lovelock [2,3] proved the following interesting theorem valid only in four dimensions.:

"If the tensor $A_i^{\ j}$ depends exclusively on the metric tensor g_{ab} and on its first and second partial derivatives

$$A_i^{\ j} = A_i^{\ j}(g_{ab}; g_{ab,c}; g_{ab,cd}), \tag{1}$$

and if it also satisfies the continuity equation

$$A_i^{\ j}{}_{;j} = 0, \tag{2}$$

then necessarily

$$A_i^{\ j} = \alpha\delta_i^{\ j} + \beta G_i^{\ j},\ \alpha, \beta = \text{constants}, \tag{3}$$

where $G_{ab} = R_{ab} - \frac{R}{2} g_{ab}$ is the Einstein tensor".

Notice that $_r$ denotes covariant derivative and besides [1] $\delta_i^{\ j}{}_{;j} = G_i^{\ j}{}_{;j} = 0$. In (1) the symmetry $A_{ij} = A_{ji}$ neither required nor needed that $A_i^{\ j}$ be linear in $g_{ab,cd}$. We do not know specific applications of this Lovelock's theorem to general relativity, except as stated by Harvey [4].

Here we shall employ (1), (2), and (3) to (a) deduce the structure of the second fundamental form of spacetimes embedded into E_5 for the case of intrinsic rigidity [1], (b) give a plain but illustrative demonstration of a Lanczos identity [5] between the curvature tensor and its double dual [1], and (c) obtain a Lanczos generator [6] in Gödel geometry [1,7].

(a) Spacetime of class one with intrinsic rigidity.

A R_4 can be embedded into E_5 if and only if there exists the second fundamental form $b_{ac} = b_{ca}$ fulfilling the Gauss-Codazzi equations [1]

$$R_{acij} = \varepsilon(b_{ai}b_{cj} - b_{aj}b_{ci}), \tag{4}$$

$$b_{ji;c} = b_{jc;i}, \tag{5}$$

where $\varepsilon = \pm 1$ and R_{acij} is the Riemann tensor, thus we say that such 4-space has class one. From the Gauss relation (4), it is possible to deduce the identity [8]

$$pb_{ij} = \frac{K_2}{48} g_{ij} - \frac{1}{2} R_{iacj} G^{ac}, \tag{6}$$

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with the presence of the Lanczos invariant [5]

$$K_2={}^*R^{*ijac}R_{ijac}\,, \tag{7}$$

in terms of the double dual [1,6] of curvature tensor ${}^*R^{*ij}{}_{ac}=\frac{1}{4}\eta^{ijrm}R_{rm}{}^{nr}\eta_{nrac}$, being η_{ijac} the Levi-Civita tensor; besides [8]

$$p^2=-\frac{\varepsilon}{6}\left(\frac{R}{24}K_2+R_{imnj}G^{ij}G^{mn}\right)\geq 0, \tag{8}$$

If $p\neq 0,$ then (6) permits to construct explicitly a b_{ij} verifying (4), and then from (6) and (8) it is clear that the intrinsic rigidity [9]

$$b_i{}^j=b_i{}^j(g_{ab};g_{ab,c};g_{ab,cd}), \tag{9}$$

If into Codazzi equation (5) we sum c with j

$$(b_i{}^j-b\delta_i{}^j)_{;j}=0\,,\ \ b=b^r{}_r\,, \tag{10}$$

then (9) and (10) imply that the tensor $A_i{}^j=b_i{}^j-b\delta_i{}^j$ satisfies the conditions (1) and (2) of the Lovelock's theorem. Thus it must have the structure (3) and therefore

$$b_{ij}=(\alpha+b)g_{ij}+\beta G_{ij}\,, \tag{11}$$

where α,β are constants. But the scalar curvature $R=-G_i{}^i$, then (11) gives us that $b=\frac{1}{3}(\beta R-4\alpha)$, thus finally (11) takes the general expression for the second fundamental form of a spacetime with intrinsic rigidity

$$b_{ij}=\beta R_{ij}-\frac{1}{6}(2\alpha+\beta R)g_{ij}\,, \tag{12}$$

such that $R_{ij}=R^a{}_{ija}$ is the Ricci tensor.

Without the Lovelock's result, it is very difficult to suspect the existence of (12). In other paper we will study the important consequences that (12) has in the local and isometric embedding of R_4 into E_5, when is present the intrinsic rigidity.

(b) A Lanczos identity

We shall employ the Lovelock's theorem to show the following Lanczos relation [5]

$${}^*R^{*jbpq}R_{ibpq}=\frac{K_2}{4}\delta_i^j\,, \tag{13}$$

which usually is proved via the generalized Kronecker´s delta [3]. However, we believe that its deduction with the aid of (1), (2) and (3) will be attractive in general relativity.

Bianchi's identities for the curvature tensor in every spacetime are [1]

$$R_{pqab;i}+R_{pqbi;a}+R_{pqia;b}=0 \tag{14}$$

or in terms of the double dual [6]

$${}^*R^{*pqab}{}_{;b}=0\,; \tag{15}$$

Besides, its known that $\eta_{pqab;c}=0$, then from (7)

$$K_{2;i}=2\,{}^*R^{*pqab}R_{pqab;i}\overset{(14)}{=}-2\,{}^*R^{*pqab}(R_{pqbi;a}+R_{pqia;b}),$$

$$=2\,{}^*R^{*pqab}(R_{ibpq;a}-R_{iapq;b}),$$

$$=4\,{}^*R^{*pqab}R_{ibpq;a}\overset{(15)}{=}(4\,{}^*R^{*abpq}R_{ibpq})_{;a}\,,$$

Thus we see that (1) and (2) are verified with $A_i{}^j={}^*R^{*jbpq}R_{ibpq}-\frac{K_2}{4}\delta_i{}^j$, then (3) implies

$${}^*R^{*jbpq}R_{ibpq}=\left(\frac{K_2}{4}+\alpha\right)\delta_i^j+\beta G_i^j\,, \tag{16}$$

whose contraction of i and j gives us

$$\beta R-4\alpha=0. \tag{17}$$

if $\beta\neq 0$ then $R=\frac{4\alpha}{\beta}=$ constant, which could be valid for some particular spacetimes; similarly $R=0$ corresponds to an specific case. But we wish a universal identity for any R_4 without restrictions on its geometry, and by (17) this is possible only if $\alpha=\beta=0$, thus (16) implies the Lanczos identity (13), q.e.d.

(c) Lanczos potential for the Gödel solution

The analysis of the invariant (7) led [6] to discover, for arbitrary spacetime, the potential K_{abc} with the properties:

$$K_{abc}=-K_{bac}\,,\; K_{abc}+K_{bca}+K_{cab}=0, \qquad (18)$$

$$K_{a\ \ r}^{\ r}=0, \qquad (19)$$

$$K_{ab\ \ ;c}^{\ \ c}=0, \qquad (20)$$

which generates the Weyl tensor through the expression [10]

$$C_{aijr}=K_{aij;r}-K_{air;j}+K_{jra;i}-K_{jri;a}+g_{ar}K_{ij}-g_{aj}K_{ir}+g_{ij}K_{ar}-g_{ir}K_{aj}, \qquad (21)$$

where $K_{ij}=K_{i\ \ j;r}^{\ r}=K_{ji}$.

Given the conformal tensor, it may be very difficult to obtain a Lanczos potential by integrating directly (21), but here we shall show that the Lovelock's result permits to determine one solution of (21) for the Gödel metric [1,7](signature +2)

$$ds^2=-(dx^1)^2-2e^{x^4}dx^1dx^2-\frac{1}{2}e^{2x^4}\left(dx^2\right)^2+\left(dx^3\right)^2+\left(dx^4\right)^2, \qquad (22)$$

with the interesting structure

$$K_{abc}=Q_{ca;b}-Q_{cb;a}\,,\; Q_{ab}=Q_{ba}, \qquad (23)$$

which verifies (18); thus the symmetric tensor Q_{ij} is a generator of the Lanczos potential.

The Lanczos algebraic gauge (19) may be satisfied if in (23) we ask the conditions

$$Q^r_{\ r}=constant, \qquad (24)$$

$$Q_{i\ \ ;j}^{\ j}=0 \qquad (25)$$

Besides, if we accept that Q_{ij} depends locally on the intrinsic geometry of R_4

$$Q_{ir}=Q_{ir}(g_{ab};g_{ab,c};g_{ab,cd}), \qquad (26)$$

then (25) and (26) imply, via the Lovelock's theorem, that

$$Q_{ab}=\alpha g_{ab}+\beta G_{ab}, \qquad (27)$$

thus $Q^r_{\ r}=4\alpha-\beta R$ is in accord with (24) because $R=1$ for the Gödel solution (22). If now we put (27) into (23), it results in

$$K_{abc}=\alpha\left(R_{ca;b}-R_{cb;a}\right), \qquad (28)$$

which also verifies the Lanczos differential gauge (20). Finally with the help of (21), (22) and (28), we conclude that $\alpha=-\frac{1}{9}$, that is,

$$K_{ijr}=\frac{1}{9}\left(R_{rj;i}-R_{ri;j}\right). \qquad (29)$$

This means that, in the Gödel cosmological model, the Ricci tensor generates one Lanczos potential.

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