# ON CERTAIN CLASS OF ANALYTIC FUNCTIONS

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**Abstract:** $P_k^{\alpha}[A,B]$ and $Q_{\alpha}^{k}[A,B]$ denote classes of functions analytic in the disc $E=\{z:|z|<1\}$ defined by a bounded radius rotation functions. In this paper we have obtained the distortion theorems, coefficients estimate, some radius problems, geometrical properties and studied convolution conditions.

***Keywords**:* Analytic, starlike, convex, positive real part function, bounded radius rotation, convolution

## Introduction

Let A denote the class of analytic functions f(z) in $E=\{z:|z|<1\}$, given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n\ , \tag{1}$$

and let S, S* and C be classes of functions in A, which are respectively univalent, starlike and convex in the unit disc E.

Janowski [4] introduced the class $P[A,B]$ as follows:

### *Definition 1*

An analytic function in E given by the form $P(z)=1+C_1Z_1+C_2Z^2+\ldots$ belongs to $P[A,B]$ if it satisfies the condition

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)},\ -1\le B<A\le 1,$$

where, w(0)=0 and |w(z)|≤1.P[1,-1]=P (the class of analytic function with positive real part satisfying Rep(z)>0).

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***Definition 2***

An analytic function in E given by (1) belongs to S*[A,B],$-1\le B<A\le 1$, if and only if, $\frac{zf'(z)}{f(z)} \in P[A,B]$ and S*[1,-1]= S*. Also it is well known that an analytic function given by (1) belongs to C[A,B] , if and only if, $\frac{(zf'(z))'}{f'(z)} \in P[A,B]$ and C[1,-1]=C.

***Definition 3***

A function f∈A is close to convex denoted by K[A,B,C,D] if ∃ a starlike function g(z)∈S*[C,D] such that $\frac{zf'(z)}{g(z)} \in P[A,B]$ and K[1,-1,1.-1]=K (the well known close to convex class due to Kaplan).

***Definition 4***

Let $P_k(\alpha), k \ge 2$ and $0<\alpha\le 1$, be the class of functions $p$ analytic in E and have the representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+(1-2\alpha)ze^{-it}}{1-ze^{-it}}\, d\mu(t) \quad ,$$

where $\mu(t)$ is a function with bounded variation on $[-\pi,\pi]$ and satisfies the conditions $\int_{-\pi}^{\pi} d\mu(t) = 2 \quad , \quad \int_{-\pi}^{\pi} |d\mu(t)| \le k$ . We note that $k \ge 2$ and $P_2(\alpha) = P[1-2\alpha,-1] = P(\alpha)$ are the class of analytic function with positive real part greater than $\alpha$ . It can easily be seen [5] that $p \in P_k(\alpha)$, if and only if, there exist two analytic functions $p_1, p_2 \in P(\alpha)$ such that

$$p(z) = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z)$$

Let $R_k(\alpha)$ denote a subclass of A of functions of bounded radius rotation of order $\alpha$ . Then f∈ $R_k(\alpha)$, if and only if,

$$\frac{zf'(z)}{f(z)} \in P_k(\alpha) \quad , \quad k \ge 2 \quad , \quad z \in E. \tag{2}$$

I t is clear that $R_2(\alpha)$ =S*(α).

Let f be given by (1) and g given by $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in$ A . Then the convolution f*g is defined by (f*g) (z) $= z+\sum_{n=2}^{\infty} a_n b_n z^n$.

***Definition 5***

Let f∈A. Then f belongs to $P_k^{\alpha}\left[A,B\right]$ if it satisfies the condition

$$\frac{f(z)}{g(z)}=\frac{1+Aw(z)}{1+Bw(z)},$$

where g ∈ $R_k(\alpha), -1\le B<A\le 1, w(z)$ is regular, w(0)=0 and w(z)≤1 and 0<α ≤1.

**Definition 6**

Let $Q_k^{\alpha}[A,B]$ denote the class of functions $F(z)=z^{-1}+c_0+c_1z+c_2z^2+\ldots$, which are regular in $0<|z|<1$ and satisfy the condition

$$\frac{F(z)}{G(z)}=\left[\frac{1+Aw(z)}{1+Bw(z)}\right]^{-1},$$

where $-1\le B<A\le 1, w(z)$ is regular in $0<|z|<1$ and $G(z)=z^{-1}+d_0+d_1z+d_2z^2+\ldots$ , is of bounded radius rotation of order α , i.e.

$$-\frac{zG'(z)}{G(z)}\in P_k(\alpha) \qquad , \qquad 0<|z|<1.$$

**Distortion theorem for the class $P_k^{\alpha}[A,B]$**

***Theorem 1***

If $f\in P_k^{\alpha}[A,B]$, then for $|z|=r$ , $0<r<1$

$$\frac{1-Ar}{1-Br}\frac{(1-r)^{(k-2)(1-\alpha)/2}}{(1+r)^{(k+2)(1-\alpha)/2}}\le|f(z)|\le\frac{1+Ar}{1+Br}\frac{(1+r)^{(k-2)(1-\alpha)/2}}{(1-r)^{(k+2)(1-\alpha)/2}} \qquad \ldots \qquad (3)$$

This result is sharp.

***Proof***

Since $f\in P_k^{\alpha}[A,B]$, we have

$$\frac{f(z)}{g(z)}=\frac{1+Aw(z)}{1+Bw(z)} \quad , \quad -1\le B<A\le 1 \quad .,$$

where g∈. $R_k(\alpha)$. By Schwarz's lemma, we have $|w(z)|\le|z|$.

If $p(z)=\dfrac{1+Aw(z)}{1+Bw(z)}$ , $-1\le B<A\le 1$ ,then it is well known [4] that p ∈ P[A,B] and satisfies

$$\frac{1-Ar}{1-Br}\le|p(z)|\le\frac{1+Ar}{1+Br} \qquad \ldots \qquad (4)$$

Further if $g(z)$ is a function of bounded radius rotation of order $\alpha$ , then by [7]

$$\frac{(1-r)^{(k-2)(1-\alpha)/2}}{(1+r)^{(k+2)(1-\alpha)/2}}\le|g(z)|\le\frac{(1+r)^{(k-2)(1-\alpha)/2}}{(1-r)^{(k+2)(1-\alpha)/2}} \qquad \ldots \qquad (5)$$

equations (4),(5) together imply the inequality (3).

This result is sharp, if we take

$$p(z)=\frac{1+Az}{1+Bz} \text{ and } g(z)=\frac{(1+\theta_1 z)^{(k-2)(1-\alpha)/2}}{(1+\theta_2 z)^{(k+2)(1-\alpha)/2}} \quad , \quad |\theta_1|=|\theta_2|=1.$$

***Remarks***

1. On taking $k=2$, we have a result of Ganesan [2].
2. On taking $k=2$, $B=-\lambda\beta$ and $A=\beta$ with $w(z)$ replaced by $-w(z)$, we get the result of Goel and Sohi [3].

**Coefficient estimates for the class $P_k^\alpha[A,B]$**

To find the coefficient estimates for the class $P_k^\alpha[A,B]$, we need the following lemmas:

***Lemma 1*** [4]

Let $p\in P[A,B]$ and $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ . Then $|c_n|\le A-B$ .

***Lemma 2***

If $p\in P_k(\alpha)$, $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ , then $|c_n|\le k(1-\alpha)$.

***Proof***

This can be easily seen using Lemma 1 and the relation

$$p(z) = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z)$$

.with A= 1- 2α and B=-1.

Using Lemma 2, we can prove Lemma 3

***Lemma 3***

Let $g \in R_k(\alpha), g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$. Then

$$|b_2| \le k(1-\alpha) \text{ and } |b_3| \le \frac{k(1-\alpha)}{2}(k - k\alpha + 1).$$

***Proof***

Let $g \in R_k(\alpha)$. Then $zg'(z) = P(z)g(z), P(z) \in P_k(\alpha)$. If $g(z) = z + b_2 z^2 + \ldots$ and $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then

$$z + 2b_2 z^2 + 3b_3 z^3 + \ldots = (z + b_2 z^2 + b_3 z^3 + \ldots)(1 + c_1 z + c_2 z^2 + \ldots)$$

Equating the coefficient of $z^2$ and $z^3$ on both sides and using Lemma1 and Lemma 2, we have

$$2b_2 = c_1 + b_2$$
$$|b_2| = |c_1| \le k(1-\alpha)$$

and $3b_3 = b_3 + c_1 b_2 + c_2$

$$|b_3| = \left|\frac{b_2 c_1 + c_2}{2}\right| \le \frac{k^2(1-\alpha)^2 + k(1-\alpha)}{2} = \frac{k(1-\alpha)}{2}(k(1-\alpha)+1).$$

***Theorem 2***

Let $f \in P_k^{\alpha}[A, B]$, where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$|a_2| \le (1-\alpha)k + (A - B)$$

and

$$|a_3| \le (A - B) + k(1-\alpha)(A - B) + \frac{k(1-\alpha)}{2}(k - k\alpha + 1)$$

These bounds are sharp.

***Proof***

Since $f \in P_k^{\alpha}[A,B]$, there exists a function $g \in R_k(\alpha)$ such that
$f(z) = g(z)p(z) \quad , \quad p \in P[A,B]$ .

If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$

and $\quad p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ ,

then

$$z + a_2 z^2 + a_3 z^3 + \ldots = (z + b_2 z^2 + b_3 z^3 + \ldots)(1 + c_1 z + c_2 z^2 + \ldots)$$

Equating the coefficient of $z^2$ and $z^3$ on both sides and using Lemma 1 and Lemma 3 we have

$$a_2 = b_2 + c_1$$
$$|a_2| = (1-\alpha)k + (A - B)$$

and
$$a_3 = c_2 + b_2 c_1 + b_3$$
$$|a_3| \leq (A-B) + k(1-\alpha)(A-B) + \frac{k(1-\alpha)}{2}(k - k\alpha + 1) .$$

This result is sharp as can be seen by the function

$$f(z) = \frac{(1-z)^{(k-2)(1-\alpha)/2}}{(1+z)^{(k+2)(1-\alpha)/2}} \frac{1+Az}{1+Bz} .$$

***Remarks***

i. If $k = 2$, this result agrees with the result of Ganesan [2] and when $k = 2$, $A = \beta$ , $B = -\lambda\beta$ , these results correspond to the result of Goel and Sohi [3].

ii. If $B = 0$, we get $\frac{f(z)}{g(z)} = 1 + Aw(z)$ and if $k = 2$ in E, the inequality $|a_n| \leq A(n-1) + n$ , $n \geq 2$ with sharp bounds as discussed in [3] is also obtainable.

**Argument of $\frac{f(z)}{z}$ when $f \in P_k^{\alpha}[A,B]$**

To discuss the argument of the class $P_k^{\alpha}[A,B]$, we need the following Lemma:

***Lemma 4***

Let $f \in R_k(\alpha)$. Then

$$\left|\arg\frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r .$$

***Proof***

It is well known that if $f \in R_k(\alpha)$, then there exist two functions $s_1, s_2 \in S^*(\alpha)$ such that

$$f(z) = \frac{(s_1(z))^{\frac{k+2}{4}}}{(s_2(z))^{\frac{k-2}{4}}} .$$

Thus
$$\left|\arg\frac{f(z)}{z}\right| = \left|\frac{k+2}{4}\arg\frac{s_2(z)}{z} - \frac{k-2}{4}\arg\frac{s_2(z)}{z}\right|$$
$$\le \frac{k+2}{4}\left|\arg\frac{s_2(z)}{z}\right| + \frac{k-2}{4}\left|\arg\frac{s_2(z)}{z}\right| .$$

It is known [8] that if $s \in S^*(\alpha)$, then

$$\left|\arg\frac{s(z)}{z}\right| \le 2(1-\alpha)\sin^{-1} r .$$

Hence
$$\left|\arg\frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r .$$

Sharpness is satisfied for $f(z) = \dfrac{(1+\theta_1 z)^{(1-\alpha)\left(\frac{k-2}{2}\right)}}{(1+\theta_2 z)^{(1-\alpha)\left(\frac{k+2}{2}\right)}}$.

***Lemma 5*** [4]

Let $p \in P[A, B]$. Then

$$\left|\arg\frac{p(z)}{z}\right| \le \sin^{-1}\frac{(A-B)r}{1-ABr^2} .$$

Using Lemma 4 and Lemma 5, we can prove

***Theorem 3***

Let $f \in P_k^{\alpha}[A, B]$. Then

$$\left|\arg \frac{f(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r + \sin^{-1}\frac{(A-B)r}{1-ABr^2}.$$

***Proof***

Since $f \in P_k^{\alpha}[A, B]$, therefore

$f(z) = g(z)p(z)$, $p(z) \in P[A, B]$ and $g \in R_k(\alpha)$. By Lemma 4, we have

$$\left|\arg \frac{g(z)}{z}\right| \le k(1-\alpha)\sin^{-1} r \qquad \dots \qquad (6)$$

and by Lemma 5, we have

$$\left|\arg p(z)\right| \le \sin^{-1}\frac{(A-B)r}{1-ABr^2} \qquad \dots \qquad (7)$$

Using (6) and (7), we have the result.

Sharpness follows by taking

$$\frac{f(z)}{g(z)} = \frac{1+A\theta_1 z}{1+B\theta_1 z} \quad , \quad |\theta_1| = 1 \qquad \dots \qquad (8)$$

and

$$g(z) = \frac{(1+\theta_2 z)^{(1-\alpha)(k-2)/2}}{(1+\theta_2 z)^{(1-\alpha)(k+2)/2}} \quad , \quad |\theta_2| = 1.$$

Then

$$\arg \frac{f(z)}{g(z)} = \sin^{-1}\frac{(A-B)r}{1-ABr^2}$$

and

$$\arg \frac{g(z)}{z} = \arg(1+\theta_2 z)^{(1-\alpha)(k-2)/2} + \arg(1+\theta_2 z)^{(1-\alpha)(k+2)/2}$$

Using Lemma 4, we have

$$\arg \frac{g(z)}{z} = (1-\alpha)k \sin^{-1} r \qquad \dots \qquad (9)$$

Using (8) and (9), we have that

$$\arg \frac{f(z)}{z} = (1-\alpha)k \sin^{-1} r + \sin^{-1} \frac{(A-B)r}{1-ABr^2}.$$

***Remark***

For $k = 2$ again this result agrees with the result in [2], and when $A = \beta > 0$, $B = -\lambda\beta$ and replacing $w(z)$ by $-w(z)$, we have the result of Goel and Sohi [3].

**Some radius problems for $P_k^{\alpha}[A,B]$**

***Lemma 6*** [1]

Let $p \in P[A,B]$. Then for $z \in E$

$$\operatorname{Re}\left\{\alpha p(z) + \beta \frac{zp'(z)}{p(z)}\right\} > \begin{cases} \dfrac{\alpha - [(A-B)\beta + 2\alpha A]r + \alpha A^2 r^2}{(1-Ar)(1-Br)} & \text{if} \quad R_1 < R_2, \\ \beta \dfrac{A+B}{A-B} + \dfrac{2}{(A-B)(1-r^2)}\left\{(L_1K_1)^{1/2} - \beta(1-ABr^2)\right\} & \text{if} \quad R_2 < R_1. \end{cases}$$

where

$$R_1 = \left(\frac{L_1}{K_1}\right)^{1/2}, R_2 = \frac{1-Ar}{1-Br}, L_1 = (1-A)(1+Ar^2) \text{ and } K_1 = (1-B)(1+Br^2)$$

This result is sharp.

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***Lemma 7*** [7]

Let g $\in R_k(\alpha)$. Then

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2}.$$

Further, since, g $\in R_k(\alpha)$ implies $\dfrac{zg'(z)}{g(z)} = f(z) \in P_k(\alpha)$, we have for all f$\in P_k(\alpha)$

$$\text{Re } f(z) \geq \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2}.$$

***Theorem 4***

Let $f \in P_k^{\alpha}[A,B]$. Then

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \begin{cases} M_1(r) & for \quad R_1 \leq R_2 \\ M_2(r) & for \quad R_2 \leq R_1 \end{cases},$$

where

$$M_1(r) = \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} - \frac{(A-B)r}{(1-Ar)(1-Br)},$$

$$M_2(r) = \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2} + \frac{A+B}{A-B} + \frac{2}{(1-r^2)(A-B)}\left[(L_1K_1)^{\frac{1}{2}1} - (1-ABr^2)\right]$$

and $R_1, R_2, L_1 and K_1$ are defined in Lemma 6 .

***Proof***

Since $f \in P_k^{\alpha}[A,B]$, there exists a function $g \in R_k(\alpha)$ such that

$$\frac{f(z)}{g(z)} = P(z) \in P[A,B]$$

Using logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}$$

and

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \min \text{Re}\frac{zg'(z)}{g(z)} + \min \text{Re}\frac{zp'(z)}{p(z)}.$$

Using Lemma 6 with $\alpha = 0$, $\beta = 1$ and Lemma 7, we have the result.

Sharpness of the bounds follow if we choose $g_i(z)(i=1,2)$, of bounded radius rotation of order α such that

***Case 1***: If $R_1 \le R_2$, we take $P_1(z) = \frac{1+Az}{1+Bz}$, and $\frac{zg_1'(z)}{g_1(z)} = \frac{1+(1-\alpha)z+(1-2\alpha)z^2}{1-r^2}$.

Then $\frac{zP_1'(z)}{P_1(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}$ .Thus at z= -r, $\text{Re}\frac{zP_1'(z)}{P_1(z)} = \frac{-(A-B)r}{(1-Ar)(1-Br)}$.

***Case 2***: If $R_2 \le R_1$, we take $p_2(z) = \frac{f_2(z)}{g_2(z)} = \frac{1+Aw_1(z)}{1+Bw_1(z)}$ and

$\frac{zg'(z)}{g(z)} = \frac{1+k(1-\alpha)w_1(z)+(1-2\alpha)w_1^{\,2}}{1-w^2(z)}$ with $w_1(z) = \frac{z(z-c_1z)}{(1-c_1z)}$, where $c_1$ defined by the condition

$\text{Re}\left[\frac{1+Aw_1(z)}{1+Bw_1(z)}\right] = R_1$ at z=-r.

Now $\frac{zp_2'(z)}{p_2(z)} = \frac{(A-B)zw_1'(z)}{(1+Aw_1(z))(1+Bw_1(z))}$.

In fact from the inequalities $R_2 \le R_1 \le c+p$, where $c = \frac{1-ABr^2}{1-B^2r^2}$, $p = \frac{(A-B)r}{1-B^2r^2}$ and we have $\frac{1-Ar}{1-Br} \le \frac{1+AT}{1+BT} \le \frac{1+Ar}{1+Br}, T = w_1(-r)$.

Hence |T|≤r and $T^2 \le r^2$ which yields

$$\frac{r^2(r+c_1)^2}{(1+rc_1)^2} \le r^2 \quad . \text{ Thus } |c_1| \le 1$$

Further $|zw_1'(z) - w_1(z)| = \frac{|z|^2 - |w(z)|^2}{1-|z|^2}$, for $w_1(z) = \frac{z(z-c_1)}{(1-c_1z)}, |c_1| \le 1$.

$$w_1(-r) = T = \frac{1-R_1}{BR_1 - A} = \frac{r(r-c_1)}{(1+c^2{}_1)}.$$

Hence $c_1 = \frac{r^2-T}{r(T-1)}$ *and* $\frac{r^2-T^2}{(1-r^2)} = \frac{r^2(1-q^2)}{(1+qr\ )^2}$ and $[zw_1'(z) - w_1(z)]_{z=1r} = \frac{r^2-T^2}{(1-r^2)}$.

Now $\mathrm{Re}\left[\frac{zp_2'(z)}{p_2(z)}\right]=\frac{(A-B)}{(1-AT)(1-BT)}\left\{T-\frac{r^2-T^2}{1-r^2}\right\}$

Using $T=\frac{1-R_1}{BR_1-A}$ with $R_1=\sqrt{\frac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)}}$ (see [1]),

and simplifying, we have $\mathrm{Re}\left[\frac{zp_2'(z)}{p_2(z)}\right]_{z=-r}=\frac{A+B}{A-B}+\frac{2}{(1-r^2)(A-B)}\left\{(L_1K_1)^{\frac{1}{2}}-(1-ABr^2)\right\}$,

where $L_1=(1-A)(1+Ar^2), K_1=(1-B)(1+Br^2)$, (see[1]).

Thus the equality in our theorem holds at z=-r for

$f_1(z)=\frac{1+Az}{1+Bz}g_1(z), if R_1\le R_2$

and for $f_2(z)=\frac{1+Aw_1(z)}{1+Bw_1(z)}g_2(z), if R_2\le R_1$ ,where $g_1(z),g_2(z)\in R_k(\alpha)$ .

***Theorem 5***

If $f\in P_k^{\alpha}[A,B]$, then $f$ is starlike in

$$|z|<\begin{cases} r_1 & for \quad R_1\le R_2 \\ r_2 & for \quad R_2\le R_1 \end{cases},$$

where $R_1$ and $R_2$ are defined as in Lemma 6 and $r_1$, $r_2$ are respectively the positive roots of the following two equations

$$(1-k(1-\alpha)r+(1-2\alpha)r^2)(1-Ar)(1-Br)-(A-B)r(1-r^2)=0$$

$$(1-k(1-\alpha)r+(1-2\alpha)r^2)(A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right]=0 \quad ,$$

where $K_1$ and $L_1$ are defined in Lemma 6. This result is sharp.

***Proof***

It follows from Theorem 4 that if $f\in P_k^{\alpha}[A,B]$, then $\mathrm{Re}\frac{zf'(z)}{f(z)}\ge M_1(r), if R_1\le R_2$ and

$\text{Re}\frac{zf'(z)}{f(z)} \geq M_2(r), if R_2 \leq R_1$. Then

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r+(1-2\alpha)r^2)(1-Ar)(1-Br)-(A-B)r(1-r^2)}{(1-r^2)(1-Ar)(1-Br)} > 0, for all |z| < r_1$$

If $R_1 \leq R_2$ and

$$\text{Re}\frac{zf'(z)}{f(z)} \geq \frac{(1-k(1-\alpha)r+(1-2\alpha)r^2)(A-B)+(1-r^2)(A+B)+2\left[(L_1K_1)^{\frac{1}{2}}-(1-ABr^2)\right]}{(1-r^2)(A-B)} > 0. \quad \text{for}$$

all, if $R_2 \leq R_1$.

For special cases see [2] and [3].

***Lemma 8***

Let $g_1(z)$ and $g_2(z) \in R_k(\alpha)$. Then $G(z) = (g_1(z))^\rho (g_2(z))^\gamma z^{1-(\rho+\gamma)}$ belongs to $R_k(\alpha_1)$ where $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

***Proof***

A logarithmic differentiation yields

$$\frac{zG'(z)}{G(z)} = \rho\frac{zg_1'(z)}{g_1(z)} + \gamma\frac{zg_2'(z)}{zg_2(z)} + (1-(\rho+\gamma))$$

$$= \rho K_1(z) + \gamma K_2(z) + (1-(\rho+\gamma))$$

where $K_1 and K_2 \in P_k(\alpha)$. From the definition of $P_k(\alpha)$, there exists $h_i, i = 1,2,3,4 \in P(\alpha)$ such that

$$\frac{zG'(z)}{G(z)} = \rho\left[\frac{k+2}{4}h_1(z) - \frac{k-2}{4}h_2(z)\right] + \gamma\left[\frac{k+2}{4}h_3(z) - \frac{k-2}{4}h_4(z)\right] + (1-(\rho+\gamma))$$

It is well known that if $h \in P(\alpha)$, then h(z) can be written as

$h(z) = (1-\alpha)p(z) + \alpha, where \text{Re}\, p(z) > 0$

and

$$\frac{zG'(z)}{G(z)} = \rho\left[\frac{k+2}{4}\left[(1-\alpha)p_1(z)+\alpha\right]\right] - \rho\,\frac{k-2}{4}\left[(1-\alpha)p_2+\alpha\right]+$$
$$\gamma\,\frac{k+2}{4}\left[(1-\alpha)p_3+\alpha\right] - \gamma\,\frac{k-2}{4}\left[(1-\alpha)p_4+\alpha\right]+\left(1-(\rho+\gamma)\right). \tag{10}$$

Since the class P is a convex set, then

$$\frac{\rho p_1(z)+\gamma p_3(z)}{\rho+\gamma} = H_1(z) \text{ and } \frac{\rho p_2(z)+\gamma p_4(z)}{\rho+\gamma} = H_2(z),$$

where $\operatorname{Re} H_I(z) > 0, i = 1,2.$ Hence(10) can be written as

$$\frac{zG'(z)}{G(z)} = \frac{k+2}{4}\left[(1-\alpha)(\rho+\gamma)H_1(z)+\left[1-(1-\alpha)(\rho+\gamma)\right]\right]$$
$$-\frac{k-2}{4}\left[(1-\alpha)(\rho+\gamma)H_2(z)+\left[1-(1-\alpha)(\rho+\gamma)\right]\right]$$
$$= \frac{k+2}{4}T_1(z)+\frac{k-2}{4}T_2(z), T_1, T_2 \in P(\alpha_1) and$$
$$\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$$

This shows that $G \in R_k(\alpha_1)$.

***Theorem 6***

Let $f_1, f_2 \in P_k^{\alpha}[A, B]$. Then

$$F(z) = (f_1(z))^{\rho}(f_2(z))^{\gamma} z^{1-(\rho+\gamma)}$$

belongs to $P_k^{\alpha_1}[A, B]$, where $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

***Proof***

Let G (z) be given by $G(z) = (g_1(z))^{\rho}(g_2(z))^{\gamma} z^{1-(\rho+\gamma)}$. Then

$$\frac{F(z)}{G(z)} = \left(\frac{f_1(z)}{g_1(z)}\right)^{\rho}\left(\frac{f_2(z)}{g_2(z)}\right)^{\gamma}$$
$$= (h_1(z))^{\rho}(h_2(z))^{\gamma} \quad , \quad (\rho+\gamma) \le 1,$$

where $h_1, h_2 \in P[A, B]$.

Hence $F \in P_k^{\alpha_1}[A,B]$, $\alpha_1 = 1-(1-\alpha)(\rho+\gamma)$.

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**Some geometrical properties**

In this part we shall investigate the behavior of $\arg f(z)$ at a point $w(\theta) = F(re^{i\theta})$ to the image $\Gamma_r$ of the circle $Cr = \{z : |z| = r\}$, $0 \le r < 1$ and where $\theta$ is any number of the interval $(0, 2\pi)$ under the mapping by means of function $f$ from the class $p_k^{\alpha}[A,B]$. We have

***Theorem 7***

If $F \in P_k^{\alpha}[A,B]$ and $0 \le r < 1$, then for $\theta_2 < \theta_1, \theta_1, \theta_2 \in [0, 2\pi]$

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) = \int_{\theta_1}^{\theta_2} \operatorname{Re}\left[\frac{re^{i\theta} f'(re^{i\theta})}{(re^{i\theta})}\right]$$

$$\ge -\pi + \{1-(1-\alpha)k + (1-2\alpha)\}(\theta_2 - \theta_1) + 2arcC\text{os}\frac{A-B}{1-AB}$$

where -1≤B<A≤1 and 0<α≤1 .

**Proof**

If $f \in P_k^{\alpha}[A,B]$, then $\dfrac{f(z)}{g(z)} = p(z)$, where $p \in P[A,B]$.

Thus

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\frac{zg'(z)}{g(z)} + \operatorname{Re}\frac{zp'(z)}{p(z)} \qquad \dots \qquad (11)$$

Let $z = re^{i\theta}$, $0 < r < 1, \theta \in [0, 2\pi]$. Integrating (11) with respect to $\theta$ in the interval $1[\theta_1, \theta_2], \theta_1 < \theta_2$, we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} d\theta = \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})$$

$$= \int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})} d\theta + \int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{re^{i\theta} p'(re^{i\theta})}{p(re^{i\theta})} d\theta$$

Since $f \in R_k(\alpha)$, it follows that

$$\min_{g \in R_k(\alpha)} \int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})} d\theta \ge \frac{1-k(1-\alpha)r + (1-2\alpha)r^2}{1-r^2}(\theta_2 - \theta_1), \quad \text{See [7].}$$

Now in the second integral, we observe that

$$\frac{\partial}{\partial\theta}\arg p(re^{i\theta}) = \frac{\partial}{\partial\theta}\operatorname{Re}\left\{-i\ln p(re^{i\theta})\right\} = \operatorname{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}.$$

Consequently

$$\int_{\theta_1}^{\theta_2}\operatorname{Re}\left[\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}\right]d\theta = \arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1})$$

and

$$\max_{p\in P[A,B]}\left|\int_{\theta_1}^{\theta_2}\operatorname{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}d\theta\right| \le \max_{p\in P[A,B]}\left|\arg p(re^{i\theta_2}) - \arg p(re^{i\theta_1})\right|$$

Using Lemma 5, we have

$$\max_{p\in P[A,B]}\arg p(re^{i\theta}) = \sin^{-1}\frac{(A-B)r}{1-ABr^2}$$

$$\begin{aligned}\max_{p\in P[A,B]}\left|\int_{\theta_1}^{\theta_2}\operatorname{Re}\frac{re^{i\theta}p'(re^{i\theta})}{p(re^{i\theta})}d\theta\right| &\le \max_{p\in P[A,B]}\left|\arg p(re^{i\theta})\right| - \min_{p\in P[A,B]}\left|\arg p(re^{i\theta})\right| \\ &\le 2\sin^{-1}\frac{(A-B)r}{1-ABr} \\ &= \pi - 2\cos^{-1}\frac{(A-B)r}{1-ABr}\end{aligned}$$

Hence

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \ge -\pi + 2Cos^{-1}\frac{(A-B)r}{1-ABr^2} + \frac{1-k(1-\alpha)r+(1-2\alpha)r^2}{1-r^2}(\theta_2-\theta_1).$$

The value of the right side is depending on the value of $r$ and it takes its smallest value at $r=1$. Thereby we obtain the required result.

**A convolution conditions for $p_k^{\alpha}[A,B]$**

In 1973, Rushweyh and Sheil-Small [9] proved the polya-Schoenberg conjecture, namely, if $f$ is convex or starlike or close to convex and $\phi$ is convex then $f*\phi$ belongs to the same class. In the following we shall prove the analogue of this conjecture for the class $p_k^{\alpha}[A,B]$ and give some of its applications. We need the following lemma with simple modification.

***Lemma 9*** [6]

Let f $\in R_k(\alpha)$. Then G = $f * \phi \in R_k(\alpha)$ where $\phi$ is convex in E

***Theorem 8***

Let $F \in P_k^{\alpha}[A,B]$ and $\phi$ is convex. Then $F * \phi \in P_k^{\alpha}[A,B]$.

***Proof:***

Let F$\in P_k^{\alpha}[A,B]$. Then F (z)=P(z) g(z) , where g belongs to $R_k(\alpha)$ and P(z)$\in$P[A,B]. It follows from the Lemma 9 that g$*\varphi \in R_k(\alpha)$ .Then $\dfrac{F*\phi}{g*\phi} \in P[A,B]$.

**Remark**

As an application of Theorem 8, we have the following

**(1)** The family $P_k^{\alpha}[A,B]$ is invariant under the following operators.

$$F_1(f) = \int_0^z \frac{f(\xi)}{\xi} d\xi = (f * \phi_1)(z)$$

$$F_2(f) = \frac{2}{z}\int_0^z f(\xi) d\xi = (f * \phi_2)(z)$$

$$F_3(f) = \int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta} d\zeta \quad , \quad |x| \le 1 \quad , \quad x \ne 1$$

$$= (f * \phi_3)(z)$$

$$F_4(f) = \frac{1+c}{c}\int_0^z \xi^{c-1} f(\xi) d\xi \quad , \quad \text{Re}\, c > 0$$

where $F(f_i(z)) = (f * \phi_i)(z)$ and $\phi_i\,(i = 1,2,3,4)$ are convex univalent functions which satisfy

$$\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z) ,$$

$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z} ,$$

$$\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x}\log\frac{1-xz}{1-z} \quad , |x| \le 1 , x \ne 1 ,$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \quad , \text{Re}\, c > 0.$$

Now let $D_\lambda F(z) = (1-\lambda)F(z) + \lambda z F'(z) = (\psi_\lambda * F)(z)$..........................................(12)

where λ>0 and let $\psi_\lambda(z) = \dfrac{z[1-(1-\lambda z)]}{1-z^2}$ . Then $\psi_\lambda(z)$ is convex if

$$|z| = r_\lambda = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}} \qquad \ldots \qquad (13)$$

Thus, we have

(2) Let $F(z) \in P_k^\alpha[A,B]$. Then $D_\lambda F(z) = \psi_\lambda * F$ belongs to the same class for $|z| < r_\lambda$, where $r_\lambda$ is given by (13).

Now let $\mu(F) = zF'(z)$. This differential operator can be written as $\mu(F) = \phi * F$,

where

$$\phi(z) = \sum_{n=1}^{\infty} n z^n = \frac{z}{1-z^2} \qquad \ldots \qquad (14)$$

It can be easily verified that the radius of convexity of $\phi$ is given by $r_c(\phi) = 2-\sqrt{3}$ . This fact together with Theorem 8 yields

(3) If $f \in P_k^\alpha[A,B]$ then $\phi * f \in P_k^\alpha[A,B]$ where $\phi$ is given by (14) if $|z| = r_c < 2-\sqrt{3}$.

**Radius of starlikeness for the class $Q_k^\alpha[A,B]$**

Now we generalize the result of Goel and Sohi [3] and Ganesen [2] for the class $Q_k^\alpha[A,B]$. The following lemma can be easily derived.

***Lemma 9***

Let $s_i, i=1,2$ be given by $s_1(z) = z^{-1} + c_\circ + c_1 z + c_2 z^2 + \ldots$ and $s_2(z) = z^{-1} + d_\circ + d_1 z + d_2 z^2 + \ldots.$ , and let $s_i, i=1,2$ satisfy $-\operatorname{Re}\dfrac{z s_i'(z)}{s_i(z)} > \alpha$ . If $G(z) = z^{-1} + b_\circ + b_1 z + b_2 z^2 + \ldots$ such that

$$G(z) = \frac{(s_1(z))^{\frac{k+2}{4}}}{(s_2(z))^{\frac{k-2}{4}}} \qquad \ldots \qquad (15)$$

then

$$-\frac{zG'(z)}{G(z)} \in P_k(\alpha) \quad .$$

***Proof***

Differentiating (15) logarithmically yields

$$\frac{zG'(z)}{G(z)} = \frac{k+2}{4}\frac{zs_1'(z)}{s_1(z)} - \frac{k-2}{4}\frac{zs_2'(z)}{s_2(z)} \quad .$$

This implies that

$$-\frac{zG'(z)}{G(z)} = \frac{k+2}{4}\left(-\frac{zs_1'(z)}{s_1(z)}\right) - \frac{k-2}{4}\left(-\frac{zs_2'(z)}{s_2(z)}\right)$$

or $$-\frac{zG'(z)}{G(z)} = \frac{k+2}{4}p_1(z) - \frac{k-2}{4}p_2(z) \ ,$$

where $\operatorname{Re} p_i(z) > \alpha$ , $i = 1, 2$ and $-\frac{zG'(z)}{G(z)} \in P_k(\alpha)$ .

***Theorem 9***

If $F \in Q_k^{\alpha}[A, B]$, then for $|z| = r < 1$

$$-\operatorname{Re}\frac{zF'(z)}{F(z)} \geq \begin{cases} \{M_1(r), \ for R_1 \leq R_2 \\ M_2(r), \ for R_2 \leq R_1 \end{cases} \quad ,$$

where

$$M_1(r) = \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1 - r^2} - \frac{(A - B)r}{(1 - Ar)(1 - Br)},$$

$$M_2(r) = \frac{1 - k(1-\alpha)r + (1-2\alpha)r^2}{1 - r^2} + \frac{A + B}{A - B} + \frac{2}{(1 - r^2)(A - B)}\left[(L_1K_1)^{\frac{1}{2}1} - (1 - ABr^2)\right]$$

and $R_1, R_2, L_1 and K_1$ are defined in Lemma 6 .

***Proof***

Since $F \in Q_k^{\alpha}[A, B]$, therefore

$$p(z)=\left[\frac{F(z)}{G(z)}\right]^{-1}=\frac{1+Aw(z)}{1+Bw(z)}\text{, where } -1\le B<A\le 1 \qquad \dots \qquad (16)$$

$w(z)$ is analytic in E and satisfies $w(0)=0$, $|w(z)|<1$ ,

Differentiating (16) logarithmically, we have

$$\frac{zp'(z)}{p(z)}=-\frac{zF'(z)}{F(z)}+\frac{zG'(z)}{G(z)}$$

or
$$-\frac{zF'(z)}{F(z)}=-\frac{zG'(z)}{G(z)}+\frac{zp'(z)}{p(z)} .$$

Using Lemma 6, we have

$$-\mathrm{Re}\frac{zF'(z)}{F(z)}\ge-\mathrm{Re}\frac{zG'(z)}{G(z)}-\frac{(A-B)r}{(1-Ar)(1-Br)} \quad if \quad R_1\le R_2$$

$$\ge-\mathrm{Re}\frac{zG'(z)}{G(z)}+\frac{(A+B)}{(A-B)}+\frac{2\left[(L_1K_1)^{1/2}-(1-ABr^2)\right]}{(A-B)(1-r^2)}$$

and since G is of bounded radius rotation of order $\alpha$ , using Lemma 7 we have

$$\mathrm{Re}-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kr+(1-2\alpha)r^2}{1-r^2}, |z|<r \qquad \dots \qquad (17)$$

Using (17), we have the required result. The bounds are sharp. This can be seen by choosing $G_1(z)$ of bounded radius variation of order $\alpha$ such that

$$-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kz+(1-2\alpha)z^2}{1-z^2} \quad if \quad R_1\ge R_2,$$

$$-\frac{zG'(z)}{G(z)}\ge\frac{1-(1-\alpha)kw_1(z)+(1-2\alpha)w_1^2(z)}{1-w_1^2(z)} \quad if \quad R_2\ge R_1$$

and take $F_1(z)$ such that it satisfies

$$p_1(z)=\left[\frac{F_1(z)}{G_1(z)}\right]^{-1}=\frac{1+Az}{1+Bz} \quad if \quad R_1\le R_2$$

$$= \frac{1 + A w_1(z)}{1 + B w_1(z)} \quad if \quad R_2 \le R_1 ,$$

where $w_1(z) = \frac{z(1 - c_1 z)}{1 - c_1 z}$ with $|c_1| \le 1$ . Proceeding in the same way as in proving the sharpness of Theorem 4, we can prove that this result is sharp.

***Theorem 10***

If $F \in Q_k^\alpha[A,B]$ , then F is starlike for $|z| = r_i < 1, i=1,2$

i. $0 < |z| < r_1 \quad for \quad R_1 \le R_2$
ii. $0 < |z| < r_2 \quad for \quad R_2 \le R_1$

where $r_1$ and $r_2$ are the smallest positive roots of the following equations respectively

$$\left[1 - k(1-\alpha)r + (1-2\alpha)r^2\right](1 - Ar)(1 - Br) - (A - B)r(1 - r^2) = 0$$

$$\left[1 - k(1-\alpha)r + (1-2\alpha)r^2\right](A - B) + (1 - r^2)(A + B) + 2\left[(L_1 K_1)^{1/2} - (1 - ABr^2)\right] = 0$$

***Proof***

Using Theorem 9, we have

$\text{Re} - \frac{zF'(z)}{F(z)} \ge M(r)_1, whenR_1 \le R_2$ and $\text{Re} - \frac{zF'(z)}{F(Z)} \ge M_2(r) whenR_2 \ge R_1$. Hence

$\text{Re} \frac{zF'(z)}{F(z)} > 0$ For $|z| < r_i, i = 1,2$ , and this gives a sufficient condition for any function F to be starlike. Proceeding in the same way as in Theorem 5, we obtain the required result.

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