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FERMIONS IN NUT-KERR-NEWMAN SPACE-TIME

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Abstract: The aim of this paper is to build up the U(1)-gauge theory for fermions in the curved space-time such as NUT-Kerr-Newman space time. The NUT-Kerr-Newman space time which is not a black hole space-time but its common feature with the black hole space-time is that it has horizon.

***Keywords*:** Gauge theory, black hole space-time, horizon, NUT parameter, null complex tetrads.

Introduction

Carmeli [1}, Carmeli and Carmeli [2]and Carmeli and Malin [3] derived the Klein-Gordon, Weyl and Dirac-type equations on $R \times S^3$ space-time by simply going from the momentum to the angular momentum representation. Sen [4] obtained the most general Lagrangians for the Dirac, Weyl, and Majorana fermions. Sen's work offers an excellent description of fermions in the space-time $R \times S^3$. M. Dariescu *et al.* [5] developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman black hole space-time. C. Dariescu et al. [6] developed the tetradic Lorentz-gauge invariant formulation of the SU(2)×U(1) theory in $R \times S^3$ space-time. Biswas [7] also developed a U(1)-gauge theory for massive fermionic fields minimally coupled to a curved space-time such as Kerr-Newman-Kasuya space-time. To develop a U(1)-gauge theory for massive fermionic fields on curved space-time such as Kerr-Newman black hole space-time they used the Dirac-type equation and the U(1)-gauge invariant Lagrangian. In this paper, we study the U(1)-gauge theory for massive fermionic fields in a NUT-Kerr-Newman space-time. The NUT-Kerr-Newman space time is not a black hole space-time but it includes Kerr-Newman black hole space-time as a special case. It also includes NUT space-time as a special case which possesses very interesting properties. Although the NUT-Kerr-Newman space-time is not a black hole space-time; its common feature with the black hole space-time is that it has horizon.

The NUT-Kerr-Newman space-time

The NUT-Kerr-Newman space-time is described by the metric

$$\begin{aligned} ds^2 &= \left(r^2 + (n + h\cos\theta)^2\right)\left(\frac{dr^2}{r^2 - 2mr + e^2 + h^2 - n^2} + d\theta^2\right) \\ &- \left(\frac{r^2 + h^2\cos^2\theta - 2mr + e^2 - n^2}{r^2 + (n + h\cos\theta)^2}\right)dt^2 \\ &+ \left\{\frac{(r^2 + h^2 + n^2)^2\sin^2\theta - (h\sin^2\theta - 2n\cos\theta)^2(r^2 - 2mr + e^2 + h^2 - n^2)}{r^2 + (n + h\cos\theta)^2}\right\}d\varphi^2 \\ &+ \frac{2\{(h\sin^2\theta - 2n\cos\theta)(r^2 - 2mr + e^2 + h^2 - n^2) - h\sin^2\theta(r^2 + h^2 + n^2)\}}{r^2 + (n + h\cos\theta)^2}\,dt\,d\varphi \end{aligned} \tag{1}$$

where m, h, e and n are the mass, angular momentum per unit mass, electric charge and NUT (magnetic mass) parameters respectively.

This is a solution to the Einstein-Maxwell equations with vector potential

$$A_\mu dx^\mu = \frac{er\{dt - (h\mathrm{Sin}^2\theta - 2n\mathrm{Cos}\theta)d\varphi\}}{r^2 + (n + h\mathrm{Cos}\theta)^2} \tag{2}$$

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The space-time given by (1) encompasses all the black hole space-times, which are asymptotically flat. Specially, the metric (1) includes:

(i) Kerr-Newman black hole space-time when $n = 0$.
(ii) Kerr black hole space-time for $n = e = 0$.
(iii) Reissner-Nordstrom black hole space-time if $n = h = 0$.
(iv) Schwarzschild black hole space-time when $n = e = h = 0$.
(v) NUT-Kerr space-time if $e = 0$.
(vi) NUT space-time with $e = h = 0$.

This metric can be transformed to Boyer coordinates under the proper coordinate transformation such as

$$\{x^{\mu}\} = \left\{ p = n + h\cos\theta \,;\, \sigma = -\frac{\varphi}{h} \,;\, q = r \,;\, \tau = t - \frac{\left(n^2 + h^2\right)}{h}\varphi \right\} \quad (3)$$

with the suitable adjustment of the parameter

$$X(p) = h^2 - (n - p)^2 \,;\, Y(q) = q^2 - 2mq + e^2 + h^2 - n^2 \quad (4)$$

Therefore, the metric (1) can be written as

$$ds^2 = \frac{p^2 + q^2}{X} dp^2 + \frac{X}{p^2 + q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2 + q^2}{Y} dq^2 - \frac{Y}{p^2 + q^2}(d\tau - p^2 d\sigma)^2 \quad (5)$$

This metric represents the NUT-Kerr-Newman space-time in Boyer coordinates, which has been studied in detail by Plebanski [8].

After a suitable choice of the null complex tetrads $\{\omega^a\}$ which consists of two complex conjugate null vectors $m, \overline{m}$ and two real null vectors k_1, k_2 ; $\{\omega^a\} = \{m, \overline{m}, k_1, k_2\}$:

$$\omega^1 = m = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2 + q^2}{X}} dp + i\sqrt{\frac{X}{p^2 + q^2}}\left(d\tau + q^2 d\sigma\right)\right]$$

$$\omega^2 = \overline{m} = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{p^2 + q^2}{X}} dp - i\sqrt{\frac{X}{p^2 + q^2}}\left(d\tau + q^2 d\sigma\right)\right] \quad (6)$$

$$\omega^3 = k_1 = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2 + q^2}}\left(d\tau - p^2 d\sigma\right) - \sqrt{\frac{p^2 + q^2}{Y}} dq\right]$$

$$\omega^4 = k_2 = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{Y}{p^2 + q^2}}\left(d\tau - p^2 d\sigma\right) + \sqrt{\frac{p^2 + q^2}{Y}} dq\right]$$

the metric (1) becomes in the simple form

$$ds^2 = 2(\omega^1\omega^2 - \omega^3\omega^4) = g_{ab}\omega^a\omega^b \quad (7)$$

with

$$\left(g_{ab}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (8)$$

Field equations

For the massive fermionic complex fields ψ the U(1)-gauge invariant Lagrangian is given by

$$L = \overline{\psi}\gamma^{\mu} D_{\mu}\psi + M\overline{\psi}\psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (9)$$

where γ^{μ} is the generalized Dirac gamma metrics, the U(1)-gauge field-strength tensor is defined as

$$F^{\mu\nu} = g^{\mu\alpha}\partial_{\alpha}A^{\nu} - g^{\nu\alpha}\partial_{\alpha}A^{\mu} - \left(g^{\mu\alpha}\partial_{\alpha}g^{\nu\beta} - g^{\nu\alpha}\partial_{\alpha}g^{\mu\beta}\right)g_{\beta\sigma}A^{\sigma} \quad (10)$$

and the gauge-covariant derivative is defined as

$$D_{\mu}\psi = \nabla_{\mu}\psi + igA_{\mu}\psi \text{ and its h.c.} \quad (11)$$

Here $\nabla_{\mu}\psi$ be the Levi-Civita covariant derivative and g be the gauge coupling constant. Under these assumptions, the Dirac-type equation is obtained in the covariant expression

$$\gamma^{\mu}\left(\partial_{\mu} + igA_{\mu}\right)\psi - \frac{1}{4}\Gamma_{\alpha\beta\mu}\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\psi + M\psi = 0 \quad (12)$$

The Dirac-type equation governs the particle, in curved space-time, and the Maxwell equations with sources can be expressed in the standard form

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}F^{\nu\mu}\right]=J^{\nu} \tag{13}$$

which will be generalized for the case of a null tetradic base $\{e_a\}$, $a=1,4$. To build up a U(1)-gauge theory of a massive fermionic complex field in the curved space-time described by the metric (7) we use the U(1)-gauge invariant Lagrangian. The general expression for the covariant derivative (11) becomes,

$$D_a\psi=\nabla_a\psi+igA_a\psi \quad \text{and its h.c.} \tag{14}$$

and the Lagrangian (9) as

$$\mathrm{L}=\bar{\psi}\gamma^{a}D_a\psi+M\bar{\psi}\psi+\frac{1}{4}F_{ab}F^{ab} \tag{15}$$

The electromagnetic tensor F^{ab} can be expressed in the base of coordinate (p,q,σ,τ)

$$F^{ab}=\omega^{a}_{\mu}\omega^{b}_{\nu}F^{\mu\nu} \tag{16}$$

Therefore, the essential components of $F^{\mu\nu}$ are given below

$$\begin{aligned}F^{12}=&\frac{X}{p^2+q^2}A^2{}_{,p}-\frac{p^2+q^2}{q^4X-p^4Y}A^1{}_{,\sigma}-\frac{p^2+q^2}{q^2X+p^2Y}A^1{}_{,\tau}\\&-\left[\frac{X\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^2\\&-\left[\frac{X(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,p}\right]A^4\end{aligned} \tag{17.1}$$

$$F^{13}=\frac{X}{p^2+q^2}A^3{}_{,p}-\frac{Y}{p^2+q^2}A_{,q}{}^1-\frac{X}{Y}\left(\frac{Y}{p^2+q^2}\right)_{,p}A^3+\frac{Y}{X}\left(\frac{X}{p^2+q^2}\right)_{,q}A^1 \tag{17.2}$$

$$\begin{aligned}F^{14}=&\frac{X}{p^2+q^2}A^4{}_{,p}-\frac{p^2+q^2}{q^2X+p^2Y}A^1{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^1{}_{,\tau}\\&-\left[\frac{X\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p}\right]A^2\\&-\left[\frac{X(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,p}+\frac{X\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,p}\right]A^4\end{aligned} \tag{17.3}$$

$$\begin{aligned}F^{23}=&-\frac{Y}{p^2+q^2}A^2{}_{,q}+\frac{p^2+q^2}{q^4X-p^4Y}A^3{}_{,\sigma}+\frac{p^2+q^2}{q^2X+p^2Y}A^3{}_{,\tau}\\&+\left[\frac{Y\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^2\\&+\left[\frac{Y(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^4X-p^4Y}\right)_{,q}\right]A^4\end{aligned} \tag{17.4}$$

$$F^{24}=\frac{p^2+q^2}{q^4X-p^4Y}A^4{}_{,\sigma}+\frac{p^2+q^2}{q^2X+p^2Y}A^4{}_{,\tau}-\frac{p^2+q^2}{q^2X+p^2Y}A^2{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^2{}_{,\tau} \tag{17.5}$$

$$\begin{aligned}F^{34}=&\frac{Y}{p^2+q^2}A^4{}_{,q}-\frac{p^2+q^2}{q^2X+p^2Y}A^3{}_{,\sigma}-\frac{p^2+q^2}{X-Y}A^3{}_{,\tau}\\&-\left[\frac{Y\left(q^4X-p^4Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X-p^2Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q}\right]A^2\\&-\left[\frac{Y(X-Y)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{X-Y}\right)_{,q}+\frac{Y\left(q^2X+p^2Y\right)}{\left(p^2+q^2\right)^2}\left(\frac{p^2+q^2}{q^2X+p^2Y}\right)_{,q}\right]A^4\end{aligned} \tag{17.6}$$

The components of $F^{\mu\nu}$ allow us to put the Maxwell equations (13) in the expression

$$e_a F^{ba}=J^b \tag{18}$$

Finally the Dirac-type equation is derived for the spinorial massive complex field ψ coupled to the NUT-Kerr-Newman-Kasuya space-time. Using the U(1) gauge invariant Lagarangian (15) the Dirac-type equation is obtained in the general form

$$\gamma^{a}\left(\partial_a+igA_a\right)\psi-\frac{1}{4}\Gamma_{bca}\gamma^{a}\gamma^{b}\gamma^{c}\psi+M\psi=0 \tag{19}$$

Hence, by working out the above, the Dirac-type equation for the metric (5) can be expressed in the form

$$\begin{aligned}&\gamma^{a}\left(\partial_a+igA_a\right)\psi+M\psi-\frac{1}{4\sqrt{2\left(p^2+q^2\right)^3}}\times\\&\left\{\left[\begin{array}{l}-\dfrac{\left(p^2+q^2\right)\left(\dfrac{\partial X}{\partial p}\right)-2pX}{\sqrt{X}}\left(\gamma^1+\gamma^2\right)\\ \qquad+\dfrac{\left(p^2+q^2\right)\left(\dfrac{\partial Y}{\partial q}\right)-2qY}{\sqrt{Y}}\left(\gamma^3-\gamma^4\right)\\ -8i\left[p\sqrt{Y}\gamma^1\gamma^2\left(\gamma^3+\gamma^4\right)-q\sqrt{X}\left(\gamma^1-\gamma^2\right)\gamma^3\gamma^4\right]\end{array}\right]\right\}\psi=0\end{aligned} \tag{20}$$

Using (3) and (4) into the equation (20), we obtain the Dirac-type equation for the metric (1) in the following form

$$\begin{aligned}&\gamma^a(\partial_a + igA_a)\psi + M\psi \\ &- \left[\frac{\left\{\left(r^2+h^2+n^2\right)h\cos\theta + nh^2\left(1+\cos^2\theta\right)\right\}\left(\gamma^1+\gamma^2\right)}{2h\sin\theta\sqrt{2\left\{r^2+\left(n+h\cos\theta\right)^2\right\}^3}}\right]\psi \\ &- \left[\frac{\left\{\left(n+h\cos\theta\right)^2(r-m)+r\left(rm-e^2-h^2+n^2\right)\right\}\left(\gamma^3-\gamma^4\right)}{2\sqrt{2\left\{r^2+\left(n+h\cos\theta\right)^2\right\}^3\left(r^2-2mr+e^2+h^2-n^2\right)}}\right]\psi \\ &+ \left[\frac{2i\left\{\left(n+h\cos\theta\right)\left(\sqrt{r^2-2mr+e^2+h^2-n^2}\right)\right\}\left(\gamma^3+\gamma^4\right)\gamma^1\gamma^2}{\sqrt{2\left\{r^2+\left(n+h\cos\theta\right)^2\right\}^3}}\right]\psi \\ &- \left[\frac{2irh\sin\theta\left(\gamma^1-\gamma^2\right)\gamma^3\gamma^4}{\sqrt{2\left\{r^2+\left(n+h\cos\theta\right)^2\right\}^3}}\right]\psi = 0\end{aligned} \tag{21}$$

In conclusion, the result obtained in this paper apply for the NUT space-time when $e = h = 0$ and for the Kerr-Newman black hole space-time when n = 0. Under this observation, we like to claim that this study not only encompasses the known result of Dariescu *et al.* [5] in the context of Kerr-Newman black hole, but also provides the similar result for the NUT space-time. So it is interesting to note that we get the U(1)-gauge theory of fermions not only in the Kerr-Newman black hole space-time[5], but also in other space-times such as Kerr-Newman-Kasuya space-time[7] and NUT-Kerr-Newman space-time which are not black hole space-times but they have the common feature with the black hole space-time that they have horizons.

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