CERTAIN SUBCLASSES OF P-VALENT STARLIKE FUNCTIONS

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Received March 2002, revised and accepted March 2006

Abstract: The object of the present paper is to introduce two interesting subclasses $T^*(p, \alpha, \beta, \gamma)$ and $C(p, \alpha, \beta, \gamma)$ of p-valent starlike functions in the open unit disc $U = \{z : |z| < 1\}$, and prove various coefficient inequalities and distortion theorems for functions belonging to these subclasses. The radii of convexity for functions belonging to the classes $T^*(p, \alpha, \beta, \gamma)$ and $C(p, \alpha, \beta, \gamma)$ are also determined.

Keywords: p-Valent, starlike, distortion theorems. AMS (2000) Subject Classification 30C50

Introduction

Let A(p) denote class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1,2....\}) \qquad (1.1)$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p-valent starlike of order α if f(z) satisfies the conditions

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (1.2)$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi \qquad (1.3)$$

for $0 \le \alpha < p, p \in N$, and $z \in U$. We denote by $S(p, \alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is called p-valent convex order α if f(z) satisfies the following conditions

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (1.4)$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \qquad (1.5)$$

for $0 \le \alpha < p, p \in N$ and $z \in U$. We denote by $K(p, \alpha)$ the class of all p-valent convex functions of order α. We note that:

$f(z) \in K(p, \alpha)$ if and only if

$$\frac{zf'(z)}{p} \in S(p, \alpha), 0 \le \alpha < p. \qquad (1.6)$$

The class $S(p, \alpha)$ was introduced by Patil and Thakare [3] and the class $K(p, \alpha)$ was introduced by Owa [2].

Let T(p) denote the subclass of A(p) consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \ge 0, p \in N). \qquad (1.7)$$

We denote by $T^*(p, \alpha)$ and $C(p, \alpha)$ the classes obtained by taking intersections, respectively, of the classes $S(p, \alpha)$ and $K(p, \alpha)$ with T(p); that is,

$$T^*(p, \alpha) = S(p, \alpha) \cap T(p)$$

and

$$C(p, \alpha) = K(p, \alpha) \cap T(p).$$

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The classes $T^*(p, \alpha)$ and $C(p, \alpha)$ were introduced by Owa [2]. In particular, the classes $S^*(1, \alpha) = S^*(\alpha)$ and $(C1, \alpha) = C(\alpha)$ when $p = 1$ were studied by Silverman [4].

Let the function $g(z)$ be defined by

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \; (b_{p+n} \geq 0, p \in N). \tag{1.8}$$

Then a function $f(z) \in T(p)$ is said to be in the class $T^*(p, \alpha, \beta, \gamma)$ if

$$\left| \frac{\frac{zf'(z)}{g(z)} - p}{\frac{zf'(z)}{g(z)} + p - 2\beta} \right| < \gamma \;\; (z \in U) \tag{1.9}$$

for $g(z) \in T^*(p, \alpha)(0 \leq \alpha < p)$, where $0 \leq \beta < p$ and $0 < \gamma \leq 1$. If a function $f(z)$ belonging to the class $T(p)$ satisfies the condition (1.9) for $g(z) \in C(p, \alpha)(0 \leq \alpha < p)$, $0 \leq \beta < p$ and $0 < \gamma \leq 1$, we say that the function $f(z)$ is in the class $C(p, \alpha, \beta, \gamma)$.

Coefficient Inequalities

We begin by recalling the following lemmas from Owa [2].

Lemma 1

Let the function $g(z)$ defined by (1.8). Then $g(z)$ is in the class $T^*(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n - \alpha) \, b_{p+n} \leq (p - \alpha). \tag{2.1}$$

Lemma 2

Let the function $g(z)$ defined by (1.8). Then $g(z)$ is in the class $C(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) \, b_{p+n} \leq p\,(p - \alpha). \tag{2.2}$$

Applying the above lemmas, we now prove our first result on the coefficient inequalities for functions belonging to the class $T^*(p, \alpha, \beta, \gamma)$, given by

Theorem 1

Let the function $f(z)$ defined by (1.7) be in the class $T^*(p, \alpha, \beta, \gamma)$. Then

$$\sum_{n=1}^{\infty} \left\{ (1+\gamma)(p+n)\, a_{p+n} - \frac{(p-\alpha)[p(1-\gamma) + 2\beta\gamma]}{(p+n-\alpha)} \right\} \leq 2\gamma(p - \beta) \tag{2.3}$$

Proof

Since $f(z) \in T^*(p, \alpha, \beta, \gamma)$, there exists a function $g(z)$ belonging to the class $T^*(p, \alpha)$ such that

$$\left| \frac{zf'(z) - pg(z)}{zf'(z) + (p - 2\beta)g(z)} \right| < \gamma \;\; (z \in U). \tag{2.4}$$

If follows from (2.4) that

$$\mathrm{Re}\left\{ \frac{\sum_{n=1}^{\infty} \{(p+n)a_{p+n} - p\sum_{n=1}^{\infty} b_{p+n}\} z^n}{2(p-\beta) - \sum_{n=1}^{\infty} \{(p+n)a_{p+n} + (p-2\beta)b_{p+n}\} z^n} \right\} < \gamma (z \in U). \tag{2.5}$$

Choose values of z on the real axis so that $\frac{zf'(z)}{g(z)}$ is real. Thus, upon clearing the denominator in (2.5) and letting $z \to 1^-$ through real values, we have

$$\sum_{n=1}^{\infty} \{(p+n)a_{p+n} - p\, b_{p+n}\} \leq$$

$$\gamma \left\{ 2(p - \beta) - \sum_{n=1}^{\infty} \{(p+n)a_{p+n} + (p - 2\beta)b_{p+n}\} \right\} \tag{2.6}$$

or, equivalently,

$$\sum_{n=1}^{\infty} \{(1+\gamma)(p+n)a_{p+n} - [p(1-\gamma) + 2\beta\gamma]b_{p+n}\} \leq 2\gamma(p - \beta). \tag{2.7}$$

Note that, by using Lemma 1, $g(z) \in T^*(p, \alpha)$ implies

$$b_{p+n} \le \frac{p-\alpha}{p+n-\alpha} \quad (n \ge 1). \qquad (2.8)$$

Making use of (2.8) in (2.7), we complete the proof of Theorem 1.

Corollary 1

Let the function f(z) defined by (1.7) be in the class T* (p, α, β, γ). Then

$$a_{p+n} \le \frac{2\gamma(p+n-\alpha)(p-\beta)+(p-\alpha)[p(1-\gamma)+2\beta\gamma]}{(p+n)(p+n-\alpha)(1+\gamma)} \quad (n \ge 1). \qquad (2.9)$$

The result (2.9) is sharp for a function of the form:

$$f(z) = z^p - \frac{2\gamma(p+n-\alpha)(p-\beta)+(p-\alpha)[p(1-\gamma)+2\beta\gamma]}{(p+n)(p+n-\alpha)(1+\gamma)} z^{p+n} \quad (n \ge 1). \qquad (2.10)$$

with respect to

$$g(z) = z^p - \frac{(p-\alpha)}{(p+n-\alpha)} z^{p+n} \ (n \ge 1). \qquad (2.11)$$

Remark 1

(i) Letting p=1 in Theorem 1 and Corollary 1, we obtain the results proved by Srivastava and Owa [5, Theorem 1 and Corollary 1].

(ii) Letting p =1 and α = 0 in Corollary 1, we obtain a result proved by Gupta [1, Theorem 3].

In a similar manner, Lemma 2 can be used to prove

Theorem 2

Let the function f(z) defined by (1.7) be in the class C(p, α, β, γ). Then

$$\sum_{n=1}^{\infty}\left\{(1+\gamma)(p+n)a_{p+n} - \frac{p(p-\alpha)[p(1-\gamma)+2\beta\gamma]}{(p+n)(p+n-\alpha)}\right\} \le 2\gamma(p-\beta). \qquad (2.12)$$

Corollary 2

Let the function f(z) defined by (1.7) be in the class C(p, α, β, γ). Then

$$a_{p+n} \le \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+p(p-\alpha)\,[p(1-\gamma)+2\beta\gamma]}{(p+n)^2(p+n-\alpha)(1+\gamma)} \quad (n \ge 1). \qquad (2.13)$$

The result (2.13) is sharp for a function of the form:

$$f(z) = z^p - \frac{2\gamma(p+n)(p+n-\alpha)(p-\beta)+p(p-\alpha)[p(1-\gamma)+2\beta\gamma]}{(p+n)^2(p+n-\alpha)(1+\gamma)} z^{p+n} \quad (n \ge 1). \qquad (2.14)$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n} \ (n \ge 1). \qquad (2.15)$$

Distortion Theorems

Applications of Lemma 1 and Lemma 2 lead to the following **distortion** theorems for functions belonging to the classes T* (p, α, β, γ) and C(p, α, β, γ).

Theorem 3

Let the function f(z) defined by (1.7) be in the class T* (p, α, β, γ). Then

$$|z|^p - A(p,\alpha,\beta,\gamma)|z|^{p+1} \le |f(z)| \le |z|^p + A(p,\alpha,\beta,\gamma)|z|^{p+1} \qquad (3.1)$$

and

$$p|z|^{p-1} - (p+1)A(p,\alpha,\beta,\gamma)|z|^p \le |f'(z)| \le p|z|^{p-1}(p+1)A(p,\alpha,\beta,\gamma)|z|^p \qquad (3.2)$$

for $z \in U$, where

$$A(p,\alpha,\beta,\gamma) = \frac{(p-\alpha)p(1+\gamma)+2\gamma(p-\beta)}{(p+1)(p+1-\alpha)(1+\gamma)}. \qquad (3.3)$$

The results (3.1) and (3.2) are sharp.

Proof

For f(z) ∈ T* (p, α, β, γ), (2.7) implies

$$(p+1)(1+\gamma)\sum_{n=1}^{\infty} a_{p+n} - [p(1-\gamma)+2\beta\gamma]\sum_{n=1}^{\infty} b_{p+n} \le 2\gamma(p-\beta). \tag{3.4}$$

For $g(z) \in T^*(p, \alpha)$, Lemma 1 implies

$$\sum_{n=1}^{\infty} b_{p+n} \le \frac{p-\alpha}{p+1-\alpha}, \tag{3.5}$$

so that (3.4) reduces to

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{(p-\alpha)p(1+\gamma)+2\gamma(p-\beta)}{(p+1)(p+1-\alpha)(1+\gamma)} = A(p,\alpha,\beta,\gamma). \tag{3.6}$$

Consequently,

$$|f(z)| \ge |z|^p - \sum_{n=1}^{\infty} a_{p+n}|z|^{p+n}$$

$$\ge |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$

$$\ge |z|^p - A(p,\alpha,\beta,\gamma)|z|^{p+1} \tag{3.7}$$

and

$$|f(z)| \le |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n}$$

$$\le |z|^p + A(p,\alpha,\beta,\gamma)|z|^{p+1}. \tag{3.8}$$

Furthermore, we note from (2.7) that

$$(1+\gamma)\sum_{n=1}^{\infty} (p+n)a_{p+n} - [p(1+\gamma)+2\beta\gamma]\sum_{n=1}^{\infty} b_{p+n} \le 2\gamma(p-\beta), \tag{3.9}$$

which, in view of (3.5), becomes

$$\sum_{n=1}^{\infty} (p+n)a_{p+n} \le \frac{(p-\alpha)p(1+\gamma)+2\gamma(p-\beta)}{(p+1-\alpha)(1+\gamma)} = (p+1)\ A(p,\alpha,\beta,\gamma). \tag{3.10}$$

Thus, we have

$$|f'(z)| \ge p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n}$$

$$\ge p|z|^{p-1} - (p+1)A(p,\alpha,\beta,\gamma)|z|^p \tag{3.11}$$

and

$$|f'(z)| \le p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n}$$

$$\le p|z|^{p-1} + (p+1)A(p,\alpha,\beta,\gamma)|z|^p. \tag{3.12}$$

Finally, we can prove that the bounds in (3.1) and (3.2) are sharp by taking the function

$$f(z) = z^p - A(p,\alpha,\beta,\gamma)z^{p+1} \tag{3.13}$$

with respect to

$$g(z) = z^p - \frac{(p-\alpha)}{(p+1-\alpha)}z^{p+1}. \tag{3.14}$$

This completes the proof of Theorem 3.

Corollary 3

Let the function f(z) defined by (1.7) be in the class $T^*(p, \alpha, \beta, \gamma)$. Then the unit disc U is mapped onto a domain that contains the disc $|w| < r_1$, where

$$r_1 = \frac{2p+1-\alpha+\gamma-\alpha\gamma+2\beta\gamma}{(p+1)(p+1-\alpha)(1+\gamma)}. \tag{3.15}$$

The result is sharp with the extremal function defined by (3.13).

Remark 2

(i) Letting p=1 in Theorem 3, we obtain a result proved by Srivastava and Owa [5, Theorem 3].

(ii) Letting p=1 and $\alpha = 0$ in Theorem 3, we obtain a result proved by Gupta [1, Theorem 4].

Theorem 4

Let the function f(z) defined by (1.7) be in the class $C(p, \alpha, \beta, \gamma)$. Then

$$|z|^p - B(p,\alpha,\beta,\gamma)|z|^{p+1} \le |f(z)| \le |z|^p + B(p,\alpha,\beta,\gamma)|z|^{p+1} \tag{3.16}$$

and

$$p|z|^{p-1} - (p+1)B(p,\alpha,\beta,\gamma)|z|^p \le |f'(z)| \le p|z|^{p-1} + (p+1)B(p,\alpha,\beta,\gamma)|z|^p \quad (3.17)$$

for $z \in U$, where

$$B(p,\alpha,\beta,\gamma) = \frac{p(p-\alpha)[p(1+\gamma)+2\beta\gamma]+2\gamma(p-\beta)(p+1)(p+1-\alpha)}{(p+1)^2(p+1-\alpha)(1+\gamma)}. \quad (3.18)$$

The results (3.16) and (3.17) are sharp.

Proof

By using Lemma 2, we have

$$\sum_{n=1}^{\infty} b_{p+n} \le \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}, \quad (3.19)$$

since $g(z) \in$ (C(p, α). The assertions (3.16) and (3.17) of Theorem 4 follow if we apply (3.19) to (2.7).

The bounds in (3.16) and (3.17) are attained by the function

$$f(z) = z^p - B(p,\alpha,\beta,\gamma)|z|^{p+1} \quad (3.20)$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} z^{p+1}. \quad (3.21)$$

This evidently completes the proof of Theorem 4.

Corollary 4

Let the function f(z) defined by (1.7) be in the class C(p, α, β, γ). Then the unit disc U is mapped onto a domain that contains the disc $|w| < r_2$, where

$$r_2 = \frac{1}{(p+1)^2(p+1-\alpha)(1+\gamma)}\{(p+1)(p+1-\alpha).$$

$$[(p+1)(1+\gamma) - 2\gamma(p-\beta] - p(p-\alpha)[p(1+\gamma)+2\beta\gamma]\}. \quad (3.22)$$

Convexity of functions in T* (p, α, β, γ) and C(p, α, β, γ)

In view of Lemma 1, we know that the function f(z) defined by (1.7) is p-valent starlike in the unit disc U if and only if

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \le p. \quad (4.1)$$

For $f(z) \in T^*(p, \alpha, \beta, \gamma)$, we find from (2.7) and (3.5) that

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \le (p+1)A(p,\alpha,\beta,\gamma) \le p, \quad (4.2)$$

where A(p, α, β, γ) is defined by (3.3). Furthermore, for $f(z) \in C(p, \alpha, \beta, \gamma)$, we

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \le (p+1)B(p,\alpha,\beta,\gamma) \le p, \quad (4.3)$$

where B(p, α, β, γ) is defined by (3.18). Thus we observe that T* (p, α, β, γ), and C(p, α, β, γ) are subclasses of p-valent starlike functions. Naturally, therefore, we are interested in finding the radii of convexity for functions in T* (p, α, β, γ) and C(p, α, β, γ). We first state:

Theorem 5

Let the function f(z) defined by (1.7) be in the class T* (p, α, β, γ) . Then f(z) is p-valent convex in the $|z| < r_3$, where

$$r_3 = \inf_{n \ge 1}\left[\frac{p^2}{(p+1)(p+n)A(p,\alpha,\beta,\gamma)}\right]^{\frac{1}{n}}$$

$$= \frac{p^2}{(p+1)^2 A(p,\alpha,\beta,\gamma)}, \quad (4.4)$$

where A(p, α, β, γ) is given by (3.3). The result is sharp.

Proof

If suffices to prove

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|\le p \quad (|z|<r_3). \tag{4.5}$$

Indeed we have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}\, z^n}{p-\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}\, z^n}\right|$$

$$\le \frac{\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}|z|^n}{p-\sum_{n=1}^{\infty}(p+n)\, a_{p+n}|z|^n}. \tag{4.6}$$

Hence (4.5) holds true if

$$\sum_{n=1}^{\infty} n(p+n)a_{p+n}|z|^n \le p^2-\sum_{n=1}^{\infty} p(p+n)a_{p+n}|z|^n, \tag{4.7}$$

that is, if

$$\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}|z|^n \le p^2. \tag{4.8}$$

With the aid of (3.10), (4.8) is true if

$$(p+n)|z|^n \le \frac{p^2}{(p+1)A(p,\alpha,\beta,\gamma)} (n\ge 1). \tag{4.9}$$

It follows from (4.9) that

$$|z|\le\left[\frac{p^2}{(p+1)(p+n)A(p,\alpha,\beta,\gamma)}\right]^{\frac{1}{n}} (n\ge 1). \tag{4.10}$$

Finally, since $(p+n)^{-n^{-1}}$ is an increasing function for integers $n\ge 1, p\in N$, we have (4.5) for $|z|<r_3$, where r_3 is given by (4.4).

In order to complete the proof of Theorem 5, we note that the result is sharp for the function f (z) e T* (p, α, β, γ) of the form:

$$f(z)=z^p-\frac{(p+1)\ A(p,\alpha,\beta,\gamma)}{(p+n)}\ z^{p+n} \tag{4.11}$$

for some integer $n\ge 1$.

Similarly, we can prove Theorem 6.

Theorem 6

Let the function f (z) defined by (1.7) be in the class C (p, α, β, γ). Then f(z) is p-valent convex in the disc $|z|<r_4$, where

$$r_4=\inf_{n\ge 1}\left[\frac{p^2}{(p+1)(p+n)B(p,\alpha,\beta,\gamma)}\right]^{\frac{1}{n}}$$

$$\frac{p^2}{(p+1)^2 B(p,\alpha,\beta,\gamma)}, \tag{4.12}$$

where B (p, α, β, γ) being given by (3.18). The result is sharp for the function $f(z)\in$ C (p, α, β, γ) of the form:

$$f(z)=z^p-\frac{(p+1)\ B(p,\alpha,\beta,\gamma)}{(p+n)}\ z^{p+n} \tag{4.13}$$

for some integer $n\ge 1$.

Acknowledgements

The authors would like to thank the referee of the paper for his helpful suggestions.

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