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ON GENERALIZATIONS OF HADAMARD PRODUCTS OF FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract: In the present paper, we define two classes of analytic functions with negative coefficients. Some interesting properties of generalizations of the Hadamard product in these classes are given.

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Introduction

Let T(n) denote the class of functions f(z) of the form

$$f(z) = z - \sum_{k=n}^{\infty} a_k z^k \,(a_k \ge 0, n \in N \setminus \{1\} = \{2,3,K\}) \quad (1.1)$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$.

A function f(z) in T(n) is said to be in the class $T_n(\lambda, \alpha)$ if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right\} > \alpha \quad (1.2)$$

for some $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$ and for all $z \in U$.

Also, let Cn (λ, α) denote the subclass of T(n) of all functions f(z) satisfying the following condition

$$\operatorname{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)}\right\} > \alpha \quad (1.3)$$

for some $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$, and for all $z \in U$.

In particular, the classes $C_2(\lambda,\alpha)$ and $T_2(\lambda,\alpha)$ were studied by Altinates and Owa [1]. Putting $\lambda=0$, we obtain the classes $T_n(0,\alpha) =: T_n(\alpha)$, and $C_n(0,\alpha) =: C_n(\alpha)$, which were investigated by Choi and Kim [2]. They are subclasses of the classes of functions starlike of order α and convex of order α, respectively, (see, for details, Duren [3] and also Srivastava and Owa [4]).

Note that

$f(z) \in C_n(\lambda, \alpha)$ if and only if $zf'(z) \in T_n(\lambda, \alpha)$.

Let $f_j(z)(j = 1,2)$ in T(n) be given by

$$f_j(z) = z - \sum_{k=n}^{\infty} a_{k,j} z^k \,(n \ge 2, j = 1,2). \quad (1.4)$$

Then the Hadamard product (or convo-lution) $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \quad (1.5)$$

For any real numbers *p* and *q*, we define the generalized Hadamard product $(f_1 \Delta f_2)$ by

$$(f_1 \Delta f_2)(p,q;z) = z - \sum_{k=n}^{\infty} (a_{k,1})^p (a_{k,2})^q z^k. \quad (1.6)$$

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In the special case, if we take p=q=1, then

$$(f_1 \Delta f_2)(1,1;z) = (f_1 * f_2)(z)(z \in U). \quad (1.7)$$

In the present paper, we make use of the generalized Hadamard product with a view to proving interesting characterization theorems involving the classes $T_n(\lambda,\alpha)$ and $C_n(\lambda,\alpha)$.

2 Main Results

In order to prove our results for functions to the general classes $T_n(\lambda,\alpha)$ and $C_n(\lambda,\alpha)$, we shall need the following Lemmas given by Altintas and Owa [1].

Lemma 1

A function f(z) defined by (1.1) is in the class if $T_n(\lambda,\alpha)$ and only if

$$\sum_{k=n}^{\infty}[k-\alpha(\lambda k+1-\lambda)]a_k \le 1-\alpha. \quad (2.1)$$

Lemma 2

A function f(z) defined by (1.1) is in the class $C_n(\lambda,\alpha)$ if and only if

$$\sum_{k=n}^{\infty}k[k-\alpha(\lambda k+1-\lambda)]a_k \le 1-\alpha. \quad (2.2)$$

Remark 1

As pointed out earlier by Altintas and Owa [1], Lemma 1 and Lemma 2 follow immediately from a result due to Altintas and Owa [1] upon setting a_k=0(k=2,3,...,n-1 (in case n=2 set $a_1 \neq 0$ because f(0)=0, f′(0)=1 which satisfy normalization).

Applying Lemma 1, we shall prove.

Theorem 1

Let functions $f_j(z)$(j=1,2) defined by (1.4) be in the classes $T_n(\lambda,\alpha_j)$ (respectively, then

$$(f_1 \Delta f_2)\left(\frac{1}{p},\frac{p-1}{p};z\right) \in T_n(\lambda,\beta_1), \quad (2.3)$$

where p > 1 and

$$\beta_1 = \min_{k\ge n}\left\{1-\frac{(k-1)(1-\lambda)}{\left(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1}\right)^{\frac{1}{p}}\left(\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2}\right)^{1-\frac{1}{p}}-1-\lambda(k-1)}\right\}. \quad (2.4)$$

Proof

Since $f_j(z) \in T_n(\lambda,\alpha_j)$, fj(z) by using Lemma 1 we have

$$\sum_{k=n}^{\infty}\left(\frac{k-\alpha_j(k\lambda+1-\lambda)}{1-\alpha_j}\right)a_{k,j} \le 1(j=1,2)$$

Moreover,

$$\sum_{k=n}^{\infty}\left(\frac{k-\alpha_1(k\lambda+1-\lambda)}{1-\alpha_1}a_{k,1}\right)^{1/p} \le 1 \quad (2.5)$$

and

$$\sum_{k=n}^{\infty}\left(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2}a_{k,2}\right)^{1/p} \le 1 \quad (2.6)$$

By the Holder inequality, we get

$$\sum_{k=n}^{\infty}\left(\frac{k-\alpha_1(k\lambda+1-\lambda)}{1-\alpha_1}\right)^{1/p}\left(\frac{k-\alpha_2(k\lambda+1-\lambda)}{1-\alpha_2}\right)^{\frac{P-1}{P}}(a_{k,1})^{1/p}(a_{k,2})^{\frac{P-1}{P}} \le 1. \quad (2.7)$$

Since

$$(f_1\Delta f_2)(\frac{1}{p},\frac{p-1}{p};z)=z-\sum_{k=n}^{\infty}(a_{k,1})^{1/p}(a_{k,2})^{p-1/p}z^{k}\le(k\ge 2) \quad (2.8)$$

we see that

$$\sum_{k=n}^{\infty}(\frac{k-\beta_1(k\lambda+1-\lambda)}{1-\beta_1}(a_{k,1})^{1/p}(a_{k,2})^{\frac{P-1}{P}}\le 1(k\ge 2) \quad (2.9)$$

with

$$\beta_1=\min_{k\ge n}\left\{1-\frac{(k-1)(1-\lambda)}{(\frac{k-\alpha_1(\lambda k+1-\lambda)}{1-\alpha_1})^{\frac{1}{p}}(\frac{k-\alpha_2(\lambda k+1-\lambda)}{1-\alpha_2})^{1-\frac{1}{p}}-1-\lambda(k-1)}\right\}.$$

Thus, by Lemma 1, the proof of Theorem 1 is complete.

Corollary 1

If the functions $f_j(z)(j=1,2)$ defined by (1.4) are in the class $T_n(\lambda,\alpha)$ T, respectively, then

$$(f_1\Delta f_2)(\frac{1}{p},\frac{p-1}{p};z)\in T_n(\lambda,\alpha) \quad (p>1). \quad (2.10)$$

Proof

In view of Lemma 1, Corollary 1 follows readily from Theorem 1 in the special case $\alpha_j=\alpha$.

Theorem 2

If the functions $f_j(z)(j=1,2$ defined by (1.4) are in the class $C_n(\lambda,\alpha j)$, respectively, then

$$(f_1\Delta f_2)(\frac{1}{p},\frac{p-1}{p};z)\in C_n(\lambda,\alpha) \quad (p>1). \quad (2.11)$$

where $p>1$ and $\beta 1$ is defined by (2.4).

Proof

Since $f_j(z)\in C_n(\lambda,\alpha_j)$, by using Lemma 2, we get

$$\sum_{k=n}^{\infty}k(\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j})a_{k,j}\le 1(j=1,2,n\ge 2). \quad (2.12)$$

Thus the proof of Theorem 2 is similar to that of Theorem 1, where Lemma 2 is used instead of Lemma 1.

Corollary 2

If the functions $f_j(z)$ $(j=1,2)$ defined by (1.4) are in the class $C_n(\lambda,\alpha)$, respectively, then

$$(f_1\Delta f_2)(\frac{1}{p},\frac{p-1}{p};z)\in C_n(\lambda,\alpha) \quad (p>1). \quad (2.13)$$

Theorem 3

Let functions $f_j(z)(j=1,2,...m)$ defined by (1.4) be in the classes $T_n(\lambda,\alpha_j)$, respectively, and let Fm(z) be defined by

$$F_m(z)=z-\sum_{k=n}^{\infty}(\sum_{j=1}^{\infty}(a_{k,j})^{p})z^{k}(p\ge 2,z\in U). \quad (2.14)$$

then

$$Fm(z)\in T_n(\lambda,\beta_n), \quad (2.15)$$

where

$$\beta_n=1-(n-1)(1-\lambda)/(\frac{1}{m}(\frac{[n-\alpha(\lambda n+1-\lambda)]}{1-\alpha})^{p}-(\lambda n+1-\lambda)), \quad (2.16)$$

$$\alpha=\min_{1\le j\le m}\alpha_j, \quad (2.17)$$

and

$$\left(\frac{n-\alpha(\lambda n+1-\lambda)}{(1-\alpha)}\right)^{p} \geq nm. \qquad (2.18)$$

Proof

Since $f_j(z)\in C_n(\lambda,\alpha_j)$, using Lemma 1, we observe that

$$\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j})a_{k,j} \leq 1(j=1,2,..,m)$$

and

$$\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j})^p(a_{k,j})^p$$
$$\leq (\sum_{k=n}^{\infty}\frac{k-\alpha_j(\lambda k+1-\lambda)}{1-\alpha_j}a_{k,j})^p \leq 1(p>1).$$

Thus we have

$$\sum_{k=n}^{\infty}\left\{\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k-\alpha_j(\lambda k+1-\lambda)}{(1-\alpha_j)}\right)^{p}(\alpha_{k,j})^{p}\right\}\leq 1.$$

Let β_k be defined by (2.16). Since

$\frac{\partial\beta_k}{\partial k} \geq 0(p\geq 2, k\geq n)$, we have

$$\beta_n \leq \beta_k\,(k\geq n). \qquad (2.19)$$

Thus by Lemma 1 and (2.17), we find that

$$\sum_{k=n}^{\infty}\frac{(k-\beta_n(\lambda k+1-\lambda)}{1-\beta_n}(\sum_{j=1}^{m}(a_{k,j})^p)$$
$$\leq\sum_{j=1}^{\infty}\frac{(k-\beta_k(\lambda k+1-\lambda)}{1-\beta_k}(\sum_{j=1}^{m}(a_{k,j})^p)$$
$$\leq \sum_{k=n}^{\infty}\frac{1}{m}\frac{(k-\alpha(\lambda k+1-\lambda)}{1-\alpha})^p(\sum_{j=1}^{m}(a_{k,j})^p)$$
$$\leq \sum_{k=n}^{\infty}\frac{1}{m}\sum_{j=1}^{\infty}(\frac{(k-\alpha(\lambda k+1-\lambda)}{1-\alpha})^p(a_{k,j})^p \leq 1. \qquad (2.20)$$

By (2.18), we see that $0\leq\beta_n<1$. Thus the proof of Theorem 3 is complete.

Theorem 4

Let functions $f_j(z)(j=1,2,...m)$ defined by (1.4) be in the class $C_n(\lambda,\alpha_j)$, respectively, and let $F_m(z)$ be defined by (2.14). Then

$$F_m(z) \in C_n(\lambda, \alpha_j) \quad (z\in U). \qquad (2.21)$$

where β_n is defined by (2.16).

Proof

Since $f_j(z) \in C_n(\lambda, \alpha_j)$, by using Lemma 2, we obtain

$$\sum_{k=n}^{\infty}\frac{k[k-\alpha_j(\lambda k+1-\lambda)]}{1-\alpha_j})a_{k,j} \leq 1 \qquad (2.22)$$

Thus the proof of Theorem 4 is analogous to Theorem 3. The details may be omitted.

Remark 2

Putting λ=0 in all results, we get the results obtained by Choi and Kim [2].

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