Some Properties of Certain New Subclasses of Analytic Functions

Rabha M. El-Ashwah¹,*, and Alaa H. Hassan²

¹Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
²Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

Abstract: In this paper, we introduce some new subclasses of analytic functions related to starlike, convex, close-to-convex and quasi-convex functions defined by using a generalized operator. Inclusion relationships for these subclasses are established. Also, we introduce some integral-preserving properties. Moreover, connections of the results presented here with those obtained in earlier works are pointed out.

Keywords: Analytic function, starlike function, convex function, close-to-convex, quasi-convex, 2010 MSC: 30C45

1. INTRODUCTION

Let \( \mathcal{A} \) denotes the class of functions \( f(z) \) which are analytic in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and be given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Let \( S \) denotes the subclass of \( \mathcal{A} \) consisting of univalent functions in \( U \). A function \( f(z) \in S \) is called starlike function of order \( \alpha (0 \leq \alpha < 1) \) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \left( 0 \leq \alpha < 1; z \in U \right),
\]

we denote the class of starlike functions by \( S^*(\alpha) \). Also, a function \( f(z) \in S \) is called convex function of order \( \alpha (0 \leq \alpha < 1) \) if and only if

\[
\Re \left\{ 1 + \frac{zf'(z)}{f(z)} \right\} > \alpha \left( 0 \leq \alpha < 1; z \in U \right).
\]

We denote this class by \( C(\alpha) \). It is well known that:

\[
f(z) \in C(\alpha) \iff zf'(z) \in S^*(\alpha) \left( 0 \leq \alpha < 1; z \in U \right).
\]

A function \( f(z) \in \mathcal{A} \) is called close-to-convex function of order \( \beta (0 \leq \beta < 1) \) and type \( \alpha (0 \leq \alpha < 1) \), if there exist a function \( g(z) \in S^*(\alpha) \) such that

Received, January 2015; Accepted, January 2016

*Corresponding author: Rabha M. El-Ashwah; E-mail: r_elashwah@yahoo.com
we denote this class by \( K(\beta, \alpha) \). Also, a function \( f(z) \in A \) is called quasi-convex function of order \( \beta(0 \leq \beta < 1) \) and type \( \alpha(0 \leq \alpha < 1) \), if there exists a function \( g(z) \in C(\alpha) \) such that

\[
Re \left\{ \frac{zf''(z)}{g(z)} \right\} > \beta(0 \leq \beta < 1; z \in U),
\]

we denote this class by \( K^*(\beta, \alpha) \). Similarly, It is well known that:

\[
f(z) \in K^*(\beta, \alpha) \iff zf''(z) \in K(\beta, \alpha) \quad (0 \leq \alpha, \beta < 1; z \in U).
\]

For further information about starlike, convex, close-to-convex and quasi-convex function, see [18], [23], [25], [27], [31], [33], [36] and [37] etc.

Following the recent work of El-Ashwah and Aouf [11] and [10], with \( p = 1 \), for \( m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \), \( \lambda > 0 \), \( l > -1 \) and \( f(z) \in A \) given by (1), the integral operator \( L_{\lambda,l}^m \) is defined as follows:

\[
L_{\lambda,l}^m f(z) = \begin{cases} f(z), & m = 0, \\ \frac{l+1}{l+\lambda(n-1)+1} \int_0^z \frac{t^{l+1-1}}{t^{l+\lambda(n-1)+1}} L_{\lambda,l}^{m-1} f(t) \, dt, & m = 1, 2, \ldots. \end{cases}
\]

It is clear from (8) that:

\[
L_{\lambda,l}^m f(z) = z + \sum_{n=2}^{\infty} \left( \frac{l+1}{l+\lambda(n-1)+1} \right)^m a_n z^n \quad (\lambda > 0; l > -1; m \in \mathbb{N}_0).
\]

Also, for \( A > 0 \) and \( a, c \in \mathbb{C} \), are such that \( Re \{c - a\} \geq 0 \), Raina and Sharma [32] defined the integral operator \( J_{a,c}^{\alpha,\beta} : A \to A \), as follows:

(i) for \( Re \{c - a\} > 0 \) and \( Re \{a\} > -A \) by

\[
J_{a,c}^{\alpha,\beta} f(z) = \frac{\Gamma(c + A)}{\Gamma(a + A) \Gamma(c - a)} \int_0^1 (1 - t)^{\alpha + 1} t^{\beta + 1} f(zt^\gamma) \, dt;
\]

(ii) for \( a = c \) by

\[
J_{a,a}^{\alpha,\beta} f(z) = f(z).
\]

For \( f(z) \) defined by (1), it is easily from (10) and (11) that:

\[
J_{a,a}^{\alpha,\beta} f(z) = z + \frac{\Gamma(c + A)}{\Gamma(a + A) \Gamma(c - a)} \sum_{n=2}^{\infty} \frac{\Gamma(a + nA)}{\Gamma(c + nA)} a_n z^n.
\]

By combining the two linear operators \( L_{\lambda,l}^m \) and \( J_{a,c}^{\alpha,\beta} \). Then, the generalized operator

\[
I_{\lambda,l}^m (a,c,A) : A \to A,
\]

is defined for the purpose of this paper as follows:

\[
I_{\lambda,l}^m (a,c,A) f(z) = L_{\lambda,l}^m \left( J_{a,a}^{\alpha,\beta} f(z) \right) = J_{a,a}^{\alpha,\beta} \left( L_{\lambda,l}^m f(z) \right),
\]

(13)
which can be easily expressed as follows:

\[ I_{\lambda,i}^m(a,c,A)f(z) = z + \frac{\Gamma(c + A)}{\Gamma(a + nA)} \sum_{n=2}^{\infty} \frac{\Gamma(a + nA)}{\Gamma(c + nA)} \left( \frac{l + 1}{l + \lambda(n - 1) + 1} \right)^n a_n z^n, \]  

\[ (A > 0; a, c \in \mathbb{C}; Re \{c - a\} \geq 0; Re \{a\} > -A; \lambda > 0; l > -1; m \in \mathbb{N}_0). \]

In view of (1.8), (1.11) and (1.13), it is clear that:

\[ I_{\lambda,i}^0(a,c,A)f(z) = J_{\lambda}^{1-c}f(z) \quad \text{and} \quad I_{\lambda,i}^m(a,a,A)f(z) = L_{\lambda,i}^m f(z). \]  

By specializing the parameters in (1.14), we note that the operator \( I_{\lambda,i}^m(a,c,A) \) generalizes a lot of previous operators, as follows:

(i) \( I_{\lambda,i}^m(\mu - 1,0,1)f(z) = I_{\lambda,i,\mu}^m f(z) \left( \lambda > 0; l > -1; \mu > 0; m \in \mathbb{N}_0 = \{ 0,1,2,... \} \right) \) (see Aouf and El-Ashwah [3]);

(ii) \( I_{\lambda,i}^m(\mu - 1,0,1)f(z) = I_{\lambda,i,\mu}^m f(z) \left( l > -1; \mu > 0; s \in \mathbb{R} \right) \) (see Cho and Kim [8]);

(iii) \( I_{\lambda,i,0}^m(\mu - 1,0,1)f(z) = I_{\lambda,i,0}^m f(z) \left( \lambda > 0; \mu > 0; m \in \mathbb{Z} \right) \) (see Aouf et al. [2]);

(iv) \( I_{\lambda,i,0}^m(a,a,A)f(z) = I_{\lambda,i,0}^m (\lambda, l) f(z) \left( \lambda > 0; l > -1; n \in \mathbb{N}_0 \right) \) (see Catas [7]);

(v) \( I_{\lambda,i}^m(a,a,A)f(z) = J_{\lambda,i}^m (\lambda, l) f(z) \left( \lambda > 0; l > -1; m \in \mathbb{N}_0 \right) \) (see El-Ashwah and Aouf [10]);

(vi) \( I_{\lambda,i,0}^m(a,a,A)f(z) = I_{\lambda,i,0}^m f(z) \left( \lambda > 0; n \in \mathbb{Z} \right) \) (see Patel [30]);

(vii) \( I_{\lambda,i,a-1}^m(a,a,A)f(z) = L_{\lambda,i}^m f(z) \left( \mu > 0; \alpha > 0 \right) \) (see Komatu [16]; Aouf [1]);

(viii) \( I_{\lambda,i,0}^m(a,a,A)f(z) = L_{\lambda,i}^m f(z) \left( \sigma > 0 \right) \) (see Jung et al. [15]; Liu [19]);

(ix) \( I_{\lambda,i}^m(a,a,A)f(z) = L_{\lambda,i}^m f(z) \left( \beta \in \mathbb{Z} \right) \) (see Uralegaddi and Somanatha [39]; Flett [12]);

(x) \( I_{\lambda,i,0}^m(a,a,A)f(z) = I_{\lambda,i,0}^m f(z) \left( n \in \mathbb{N}_0 \right) \) and \( I_{\lambda,i,0}^m(a,a,A)f(z) = D_{\lambda,i}^m f(z) \left( n \in \mathbb{N}_0 \right) \) (see Salagean [35]);

(xi) \( I_{\lambda,i}^m(a,a,A)f(z) = P_{\lambda,i}^m f(z) \left( \mu > 0; l > -1 \right) \) (see Gao et al. [13]);

(xii) \( I_{\lambda,i,0}^m(a,a,A)f(z) = L_{\alpha,i}^m f(z) \left( \sigma > 0 \right) \) (see Owa and Srivastava [29]; Srivastava and Owa [38]);

(xiii) \( I_{\lambda,i}^m(\beta, \alpha + \beta - \gamma + 1, 1)f(z) = \mathcal{R}_{\beta,i}^m f(z) \left( \gamma > 0; \alpha \geq \gamma - 1; \beta > -1 \right) \) (see Aouf et al. [4]);

(xiv) \( I_{\lambda,i}^m(\beta, \alpha + \beta, 1)f(z) = Q_{\beta,i}^m f(z) \left( \alpha \geq 0; \beta > -1 \right) \) (see Liu and Owa [21]; Jung et al. [15]; Li [17]; Liu [20]);

(xv) \( I_{\lambda,i}^m(a, 1-c, -1, 1, 1)f(z) = L(a, c) f(z) \left( a, c \in \mathbb{C} \setminus \mathbb{Z}_0, \mathbb{Z}_0 = \{ 0, -1, -2, ... \} \right) \) (see Carlson and Shaffer [6]);

(xvi) \( I_{\lambda,i}^m(\mu - 1, 1, v, 1)f(z) = I_{\lambda,i,v}^m f(z) \left( \mu > 0; v > -1 \right) \) (see Choi et al. [9]);

(xvii) \( I_{\lambda,i}^m(\alpha, 0, 1)f(z) = D_{\alpha,i}^m f(z) \left( \alpha > -1 \right) \) (see Ruscheweyh [34]);

(xviii) \( I_{\lambda,i}^m(1, n, 1)f(z) = D_{\lambda,i}^m f(z) \left( n \in \mathbb{N}_0 \right) \) (see Noor [26]; Noor and Noor [28]).

Using (1.14), we can obtain the following recurrence relations, which are needed for our proofs in following two sections:

\[ z \left( I_{\lambda,i}^{m+1}(a,c,A)f(z) \right)^{'} = \frac{1 + l}{\lambda} I_{\lambda,i}^m(a,c,A)f(z) - \frac{1 + l - \lambda}{\lambda} I_{\lambda,i}^{m+1}(a,c,A)f(z), \]  

\[ \]
\begin{equation}
    z \left( I_{a,l}^m (a,c,A) f(z) \right)' = \frac{a+A}{A} I_{a,l}^m (a+1,c,A) f(z) - \frac{a}{A} I_{a,l}^m (a,c,A) f(z).
    \tag{17}
\end{equation}

**Definition 1.** By using the operator $I_{a,l}^m (a,c,A) f(z)$ defined by (14), we introduce the following subclasses of the class $A$, as follows:

- $S_{a,l}^m (a,c,A;\alpha) = \left\{ f : f(z) \in A \text{ and } I_{a,l}^m (a,c,A) f(z) \in S^+(\alpha) \right\}$, \tag{18}
- $C_{a,l}^m (a,c,A;\alpha) = \left\{ f : f(z) \in A \text{ and } I_{a,l}^m (a,c,A) f(z) \in C(\alpha) \right\}$, \tag{19}
- $K_{a,l}^m (a,c,A;\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } I_{a,l}^m (a,c,A) f(z) \in K(\beta,\alpha) \right\}$, \tag{20}
- $K_{a,l}^m (a,c,A;\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } I_{a,l}^m (a,c,A) f(z) \in K^-(\beta,\alpha) \right\}$. \tag{21}

$(A > 0, a,c \in \mathbb{C}, Re \{c-a\} \geq 0, Re \{a\} > -A, \lambda > 0, l > -1, 0 \leq \alpha, \beta < 1, m \in \mathbb{N}_0)$.

**Remark 1.** If we set $a = c$ in Definition 1, we obtain the following subclasses of $A$:

- $S_{a,l}^m (\alpha) = \left\{ f : f(z) \in A \text{ and } L_{a,l}^m f(z) \in S^+(\alpha) \right\}$, \tag{22}
- $C_{a,l}^m (\alpha) = \left\{ f : f(z) \in A \text{ and } L_{a,l}^m f(z) \in C(\alpha) \right\}$, \tag{23}
- $K_{a,l}^m (\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } L_{a,l}^m f(z) \in K(\beta,\alpha) \right\}$, \tag{24}
- $K_{a,l}^m (\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } L_{a,l}^m f(z) \in K^-(\beta,\alpha) \right\}$. \tag{25}

Where $L_{a,l}^m f(z)$ is defined by (9).

**Remark 2.** If we set $m = 0$ in Definition 1, we obtain the following subclasses of $A$:

- $S^+(a,c,A;\alpha) = \left\{ f : f(z) \in A \text{ and } J_{a}^{+c} f(z) \in S^+(\alpha) \right\}$, \tag{26}
- $C(a,c,A;\alpha) = \left\{ f : f(z) \in A \text{ and } J_{a}^{+c} f(z) \in C(\alpha) \right\}$, \tag{27}
- $K(a,c,A;\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } J_{a}^{+c} f(z) \in K(\beta,\alpha) \right\}$, \tag{28}
- $K^+(a,c,A;\beta,\alpha) = \left\{ f : f(z) \in A \text{ and } J_{a}^{+c} f(z) \in K^+(\beta,\alpha) \right\}$. \tag{29}

Where $J_{a}^{+c} f(z)$ is defined by (12).

In order to introduce our main results, we shall need the following lemma which is given by Miller and Mocanu [24].

**Lemma 1.** Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and let $\psi(u,v)$ be a complex-valued function such that $\psi : D \to \mathbb{C}$ $(D \subset \mathbb{C} \times \mathbb{C})$.

Suppose also that the function $\psi(u,v)$ satisfies each of the following conditions:

(i) $\psi(u,v)$ is continuous in $D$;

(ii) $(1,0) \in D$ and $Re \{\psi(1,0)\} > 0$;

(iii) $Re \{\psi(iu_2,v_1)\} \leq 0$ for all $(iu_2,v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$. \tag{30}
Let
\[ h(z) = 1 + c_1 z + c_2 z^2 + \ldots, \]
be analytic in \( U \) such that \((h(z), zh'(z)) \in D \ (z \in U)\). If \( \Re \{\psi(h(z), zh'(z))\} > 0 \ (z \in U) \), then \( \Re \{h(z)\} > 0 \) for \( z \in U \).

2. INCLUSION RELATIONSHIPS

Unless otherwise mentioned, we shall assume throughout the paper that \( A > 0, a, c \in \mathbb{C}, \Re \{c - a\} \geq 0, \Re \{a\} > -A, \lambda > 0, l > -1, 0 \leq \alpha, \beta < 1, m \in \mathbb{N}_0 \) and \( f(z) \in \mathcal{A} \).

In this section, we give several inclusion relationships for analytic function classes, which are associated with the generalized operator \( I_{a,l}^n (a,c,A)f(z) \) defined by (14).

**Theorem 1.** Let \( a = a_1 + ia_2 \) with \( \Re \{a\} > -A \alpha \), then
\[ S_{a_1}^m (a + 1,c,A;\alpha) \subset S_{a_2}^m I_{a_1}^n(a,c,A;\alpha) \subset S_{a_1}^m (a,c,A;\alpha). \] (32)

Proof. (i) we begin with showing the first inclusion relationship
\[ S_{a_2}^m (a + 1,c,A;\alpha) \subset S_{a_1}^m (a,c,A;\alpha), \] (33)
which is asserted by Theorem 1. Let \( f(z) \in S_{a_2}^m (a + 1,c,A;\alpha) \) and set
\[ \frac{z}{I_{a_1}^n (a,c,A)f(z)} - \alpha = (1-\alpha)h(z), \] (34)
where \( h(z) \) is defined by (31). By using the identity (17) and (34), we obtain
\[ \frac{a + A}{A} I_{a_1}^n (a + 1,c,A)f(z) = (1-\alpha)h(z) + \alpha + \frac{A}{a}. \] (35)

By using logarithmic differentiation on both sides of (35), we obtain
\[ \frac{z}{I_{a_1}^n (a + 1,c,A)f(z)} \frac{\left(I_{a_1}^n (a + 1,c,A)f(z)\right)'}{I_{a_1}^n (a + 1,c,A)f(z)} = \frac{z}{I_{a_1}^n (a,c,A)f(z)} \frac{\left(I_{a_1}^n (a,c,A)f(z)\right)'}{I_{a_1}^n (a,c,A)f(z)} + \frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z) + \alpha + \frac{A}{a}}, \]
using (34) in the above equation, we obtain
\[ \frac{z}{I_{a_1}^n (a + 1,c,A)f(z)} \frac{\left(I_{a_1}^n (a + 1,c,A)f(z)\right)'}{I_{a_1}^n (a + 1,c,A)f(z)} - \alpha = (1-\alpha)h(z) + \frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z) + \alpha + \frac{A}{a}}. \]

or
\[ \frac{z}{I_{a_1}^n (a + 1,c,A)f(z)} \frac{\left(I_{a_1}^n (a + 1,c,A)f(z)\right)'}{I_{a_1}^n (a + 1,c,A)f(z)} - \alpha = (1-\alpha)h(z) + \frac{A (1-\alpha)zh'(z)}{A [(1-\alpha)h(z) + \alpha + a] + a}. \] (36)

Now, we choose \( u = h(z) = u_1 + iu_2 \) and \( v = zh'(z) = v_1 + iv_2 \), and define the function \( \psi(u,v) \) by
\[ \psi(u,v) = (1-\alpha)u + \frac{A (1-\alpha)v}{A [(1-\alpha)u + \alpha] + a}. \]
It is easy to see that the function \( \psi(u,v) \) satisfies conditions (i) and (ii) of Lemma 1 in 
\( D = \left( \mathbb{C} \setminus \left\{ \frac{a_1 + ia_2}{d(\alpha - 1)} \right\} \right) \times \mathbb{C} \). Also, we verify condition (iii) as follows:

\[
\text{Re}\left\{ \psi(iu_2, v) \right\} = \text{Re}\left\{ \frac{A(1-\alpha)v_i}{A[(1-\alpha)u_2 + \alpha] + a} \right\} = \frac{A(\alpha A + a_i)(1-\alpha)v_i}{[(\alpha A + a_i)^2 + (A(1-\alpha)u_2 + a)^2]} \leq \frac{A(\alpha A + a_i)(1-\alpha)(1+u_2^2)}{2[(\alpha A + a_i)^2 + (A(1-\alpha)u_2 + a)^2]} < 0.
\]

Which shows that \( \psi(u,v) = \psi(h(z), zh'(z)) \) \((z \in U)\) satisfies the hypotheses of the Lemma 1, then \( \text{Re}\{h(z)\} > 0 \) \((z \in U)\), then using (34), we have \( f(z) \in S_{\lambda,l}^m(a,c,A;\alpha) \). This completes the proof of (33).

(ii) Now, we prove

\[
S_{\lambda,l}^m(a,c,A;\alpha) \subset S_{\lambda,l}^{m+1}(a,c,A;\alpha),
\]

which is the second inclusion relationship of Theorem 1. Let \( f(z) \in S_{\lambda,l}^m(a,c,A;\alpha) \) and set

\[
\frac{z}{I_{\lambda,l}^m(a,c,A)f(z)}' - \alpha = (1-\alpha)h(z),
\]

where \( h(z) \) is defined by (31). By applying the identity (16) in (38), we obtain

\[
\frac{1+l}{\lambda} I_{\lambda,l}^m(a,c,A)f(z) = (1-\alpha)h(z) + \alpha + \frac{1+l-\lambda}{\lambda}.
\]

By using logarithmic differentiation on both sides of (39), we obtain

\[
\frac{z}{I_{\lambda,l}^m(a,c,A)f(z)}' - \frac{z}{I_{\lambda,l}^{m+1}(a,c,A)f(z)}' = \frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z) + \alpha + \frac{1+l-\lambda}{\lambda}},
\]

using (38) in the above equation, we have

\[
\frac{z}{I_{\lambda,l}^m(a,c,A)f(z)}' - \alpha = (1-\alpha)h(z) + \frac{\lambda (1-\alpha)zh'(z)}{\lambda (1-\alpha)h(z) + \alpha + (1+l-\lambda)}.
\]

By using arguments similar to those detailed before, together with (40) and \( \psi(u,v) \) is continuous in 
\( D = \left( \mathbb{C} \setminus \left\{ \frac{a_1 + ia_2}{d(\alpha - 1)} \right\} \right) \times \mathbb{C} \), then we can prove (37), which is the second inclusion relationship of Theorem 1. Combining the inclusion relationships (33) and (37), we complete the proof of Theorem 1.

**Theorem 2.** Let \( a = a_1 + ia_2 \) with \( \text{Re}\{a\} > -A\alpha \), then

\[
C_{\lambda,l}^m(a+1,c,A;\alpha) \subset C_{\lambda,l}^m(a,c,A;\alpha) \subset C_{\lambda,l}^{m+1}(a,c,A;\alpha).
\]

**Proof.** We first show that

\[
C_{\lambda,l}^m(a+1,c,A;\alpha) \subset C_{\lambda,l}^m(a,c,A;\alpha).
\]

Let \( f(z) \in C_{\lambda,l}^m(a+1,c,A;\alpha) \). Then, using Theorem 1, we have
Some Properties of Certain New Subclasses of Analytic Functions

\[ I_{\lambda,\beta}^a (a + 1, c, \lambda) f (z) \in C (\alpha) \quad \Leftrightarrow \quad z \left( I_{\lambda,\beta}^a (a + 1, c, \lambda) f (z) \right) ' \in S^* (\alpha) \]

\[ I_{\lambda,\beta}^a (a + 1, c, \lambda) z f ' (z) \in S^* (\alpha) \]

\[ zf ' (z) \in S_{\lambda,\beta}^a (a, c, \lambda; \alpha) \]

\[ z f ' (z) \in S_{\lambda,\beta}^a (a, c, \lambda; \alpha) \]

\[ I_{\lambda,\beta}^a (a, c, \lambda) z f ' (z) \in S^* (\alpha) \]

\[ z \left( I_{\lambda,\beta}^a (a, c, \lambda) f (z) \right) ' \in S^* (\alpha) \]

\[ I_{\lambda,\beta}^a (a, c, \lambda) f (z) \in C (\alpha) \]

\[ f (z) \in C_n^a (a, c, \lambda; \alpha). \]

This completes the proof of (42). By using arguments similar to those detailed above, we can also prove the right part of Theorem 2, that is, that

\[ C_{\lambda,\beta}^n (a, c, \lambda; \alpha) \subset C_{\lambda,\beta}^{n+1} (a, c, \lambda; \alpha). \quad \text{(43)} \]

Combining the inclusion relationships (42) and (43), then the proof of Theorem 2 is completed.

**Theorem 3.** Let \( a = a_1 + a_2 \) with \( Re \{ a \} > -A \alpha \), then

\[ K_{\lambda,\beta}^n (a + 1, c, \lambda; \beta, \alpha) \subset K_{\lambda,\beta}^n (a, c, \lambda; \beta, \alpha) \subset K_{\lambda,\beta}^{n+1} (a, c, \lambda; \beta, \alpha). \quad \text{(44)} \]

Proof. Let us begin with proving that

\[ K_{\lambda,\beta}^n (a + 1, c, \lambda; \beta, \alpha) \subset K_{\lambda,\beta}^n (a, c, \lambda; \beta, \alpha). \quad \text{(45)} \]

Let \( f (z) \in K_{\lambda,\beta}^n (a + 1, c, \lambda; \beta, \alpha) \). Then there exists a function \( g (z) \in S^* (\alpha) \) such that

\[ Re \left\{ \frac{z \left( I_{\lambda,\beta}^a (a + 1, c, \lambda) f (z) \right) '}{g (z)} \right\} > \beta (z \in U). \]

We put \( g (z) = I_{\lambda,\beta}^a (a + 1, c, \lambda) k (z) \), so that we have \( k (z) \in S_{\lambda,\beta}^a (a + 1, c, \lambda; \alpha) \) and

\[ Re \left\{ \frac{z \left( I_{\lambda,\beta}^a (a + 1, c, \lambda) f (z) \right) '}{I_{\lambda,\beta}^a (a + 1, c, \lambda) k (z)} \right\} > \beta (z \in U). \]

Next, we put

\[ \frac{z \left( I_{\lambda,\beta}^a (a, c, \lambda) f (z) \right) '}{I_{\lambda,\beta}^a (a, c, \lambda) k (z)} = (1 - \beta) h (z) + \beta, \quad \text{(46)} \]

where \( h (z) \) is given by (31). Thus, by using the identity (17), we obtain

\[ \frac{z \left( I_{\lambda,\beta}^a (a + 1, c, \lambda) f (z) \right) '}{I_{\lambda,\beta}^a (a + 1, c, \lambda) k (z)} = I_{\lambda,\beta}^a (a + 1, c, \lambda) z f ' (z) \]

\[ I_{\lambda,\beta}^a (a + 1, c, \lambda) k (z) \]

\[ = \frac{a}{A + a} z \left( I_{\lambda,\beta}^a (a, c, \lambda) z f ' (z) \right) + \frac{a}{A + a} I_{\lambda,\beta}^a (a, c, \lambda) z f ' (z) \]

\[ = \frac{a}{A + a} z \left( I_{\lambda,\beta}^a (a, c, \lambda) k (z) \right) + \frac{a}{A + a} I_{\lambda,\beta}^a (a, c, \lambda) k (z) \]

\[ A z \left( I_{\lambda,\beta}^a (a, c, \lambda) f (z) \right) \]

\[ = A z \left( I_{\lambda,\beta}^a (a, c, \lambda) k (z) \right) \]

\[ + a \left( I_{\lambda,\beta}^a (a, c, \lambda) k (z) \right) \]

\[ = A \left( I_{\lambda,\beta}^a (a, c, \lambda) k (z) \right) + a \left( I_{\lambda,\beta}^a (a, c, \lambda) k (z) \right). \quad \text{(47)} \]
Since \( k(z) \in S_{\lambda,l}^{m}(a+1,c,A;\alpha) \), by using Theorem 1, we can put
\[
\frac{z \left( I_{\lambda,l}^{m}(a,c,A)k(z) \right)'}{I_{\lambda,l}^{m}(a,c,A)k(z)} = (1-\alpha)G(z) + \alpha,
\]
(48)
where \( G(z) = g_1(x,y) + ig_2(x,y) \) and \( \text{Re}\{G(z)\} = g_1(x,y) > 0 \) \((z \in U)\). Using (46) and (48) in (47), we have
\[
\frac{z \left( I_{\lambda,l}^{m}(a+1,c,A)f(z) \right)'}{I_{\lambda,l}^{m}(a+1,c,A)k(z)} = A \frac{z \left[ I_{\lambda,l}^{m}(a,c,A)f'(z) \right]'}{I_{\lambda,l}^{m}(a,c,A)k(z)} + a \left[ \beta + (1-\beta)h(z) \right] + \alpha.
\]
(49)
Moreover, from (46), we can put
\[
z \left( I_{\lambda,l}^{m}(a,c,A)f(z) \right)' = \left[ (1-\beta)h(z) + \beta \right] I_{\lambda,l}^{m}(a,c,A)k(z).
\]
(50)
Differentiating both sides of (50) with respect to \( z \), and using (46) and (48), we obtain
\[
z \left( I_{\lambda,l}^{m}(a,c,A)f'(z) \right) - \beta = (1-\beta)h(z) + \beta \left[ (1-\alpha)G(z) + \alpha \right].
\]
(51)
By substituting (51) into (49), we obtain
\[
z \left( I_{\lambda,l}^{m}(a+1,c,A)f(z) \right)' - \beta = (1-\beta)h(z) + \frac{A (1-\beta)h'(z)}{A \left[ (1-\alpha)G(z) + \alpha \right] + a}.
\]
(52)
In (52), take \( u = h(z) = u_1 + iu_2 \), \( v = h'(z) = v_1 + iv_2 \) and define the function \( \psi(u,v) \) by
\[
\psi(u,v) = (1-\beta)u + \frac{A (1-\beta)v}{A \left[ (1-\alpha)G(z) + \alpha \right] + a},
\]
(53)
where \((u,v) \in D = \mathbb{C} \times \mathbb{C}\). Then it follows from (53) that:
\(i\) \( \psi(u,v) \) is continuous in \( D \);
\(ii\) \( (1,0) \in D \) and \( \text{Re}\{\psi(1,0)\} = 1-\beta > 0 \);
\(iii\) for all \((iu_z,v_z) \in D\) such that \( v_z \leq -\frac{1}{2} \left( 1 + u_z^2 \right) \), we have
\[
\text{Re}\{\psi(iu_z,v_z)\} = \text{Re}\left\{ \frac{A (1-\beta)v_z}{A \left[ (1-\alpha)G(z) + \alpha \right] + a} \right\}
\]
\[
= \left( A (1-\beta)v_z \right) \left[ \left( A \left[ (1-\alpha)g_1(z) + \alpha \right] + a \right) \right]
\]
\[
= \left( A \left[ (1-\alpha)g_1(z) + \alpha \right] + a \right) \left( A \left[ (1-\alpha)g_1(z) + \alpha \right] + a \right)
\]
\[
- \left( A (1-\beta) \left( 1 + u_z^2 \right) \right) \left[ \left( A \left[ (1-\alpha)g_1(z) + \alpha \right] + a \right) \right]
\]
\[
< 0,
\]
which shows that \( \psi(u,v) \) satisfies the hypotheses of Lemma 1. Thus, in light of (46), we easily deduce the inclusion relationship (45).
The remainder of our proof of Theorem 3 would make use of the identity (16) in an analogous manner. Therefore, we choose to omit the details involved.

**Theorem 4.** Let \( a = a_1 + ia_2 \) with \( \text{Re}\{a\} > -A \alpha \), then

\[
K_{\lambda,i}^m (a+1, c, A; \beta, \alpha) \subset K_{\lambda,i}^{*m} (a, c, A; \beta, \alpha) \subset K_{\lambda,i}^{m+1} (a, c, A; \beta, \alpha). \tag{54}
\]

Proof. Just, as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (4). Similarly, we can prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (7). Therefore, we choose to omit the details involved.

**Remark 3.**

(i) Taking \( m = s (s \in \mathbb{R}) \), \( \lambda = 1 \), \( a = \mu - 1 (\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 1-3, we obtain the results obtained by Cho and Kim [8, Theorems 2.1-2.3 with \( \varphi(z) = \psi(z) = \frac{1 - z}{1 + z} \)];

(ii) Taking \( a = \mu - 1 (\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 1-3, we obtain the results obtained by Aouf and El-Ashwah [3, Theorems 1-3 with \( \varphi(z) = \psi(z) = \frac{1 - z}{1 + z} \)];

(iii) Taking \( l = 0 \), \( a = \mu - 1 (\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 1-3, we obtain the results obtained by Aouf et al. [2, Theorems 1-3 with \( \varphi(z) = \psi(z) = \frac{1 - z}{1 + z} \)].

Taking \( a = c \) in Theorems 1-4, we obtain the following corollary.

**Corollary 1.** For the subclasses \( S_{\lambda,i}^{*m} (\alpha) \), \( C_{\lambda,i}^{*m} (\alpha) \), \( K_{\lambda,i}^{*m} (\beta, \alpha) \) and \( K_{\lambda,i}^{m+1} (\beta, \alpha) \) defined in Remark 1, we have the following inclusion relations.

\[
S_{\lambda,i}^{*m} (\alpha) \subset S_{\lambda,i}^{m+1} (\alpha),
C_{\lambda,i}^{*m} (\alpha) \subset C_{\lambda,i}^{m+1} (\alpha),
K_{\lambda,i}^{*m} (\beta, \alpha) \subset K_{\lambda,i}^{m+1} (\beta, \alpha),
K_{\lambda,i}^{*m} (\beta, \alpha) \subset K_{\lambda,i}^{*m+1} (\beta, \alpha).
\]

**Remark 4.**

(i) Taking \( \lambda = 1 \), \( m = \mu (\mu > 0) \) and \( l = a - 1 (a > 0) \) in Corollary 1, we obtain the results obtained by Aouf [1, Theorems 1-4];

(ii) Taking \( \lambda = l = 1 \) and \( m = \sigma (\sigma > 0) \) in Corollary 1, we obtain the results obtained by Liu [19, Theorems 1-4].

Taking \( m = 0 \) in Theorems 1-4, we obtain the following corollary.

**Corollary 2.** For the subclasses \( S^* (a, c, A; \alpha) \), \( C (a, c, A; \alpha) \), \( K (a, c, A; \beta, \alpha) \) and \( K^* (a, c, A; \beta, \alpha) \) defined in Remark 2, we have the following inclusion relations.

\[
S^* (a+1, c, A; \alpha) \subset S^* (a, c, A; \alpha),
C (a+1, c, A; \alpha) \subset C (a, c, A; \alpha),
K (a+1, c, A; \beta, \alpha) \subset K (a, c, A; \beta, \alpha),
K^* (a+1, c, A; \beta, \alpha) \subset K^* (a, c, A; \beta, \alpha).
\]
Remark 5. Taking $\alpha = \beta = 0$, $a = \mu - 1 (\mu > 0)$, $c = \lambda (\lambda > -1)$ and $A = 1$ in the first three inclusion relationships of Corollary 2, we obtain the results obtained by Choi et al. [9, Theorems 1-3, only the first parts with $\varphi(z) = \psi(z) = \frac{1 + \varphi(z)}{1 - z}$].

3. INTEGRAL-PRESERVING PROPERTIES INVOLVING THE INTEGRAL OPERATOR $L_\sigma$

For $\sigma > -1$ and $f(z) \in A$, we recall the generalized Bernardi-Libera-Livingston integral operator $L_\sigma : A \to A$, as following (see Owa and H. M. Srivastava [29]):

$$L_\sigma f(z) = \frac{\sigma + 1}{z^\sigma} \int_0^z f(t) t^{\sigma-1} dt$$

$$z + \sum_{n=\sigma}^\infty \left( \frac{\sigma + 1}{\sigma + n} \right) a_n z^n \quad (\sigma > -1, f(z) \in A). \quad (55)$$

The operator $L_\sigma f(z)$ was introduced by Bernardi [5]. In particular, the operator $L_1 f(z)$ was studied earlier by Libera [18] and Livingston [22]. Using (14) and (55), it is clear that $L_\sigma f(z)$ satisfies the following relationship:

$$z \left[ I_{\lambda,1}^m (a,c,A) L_\sigma f(z) \right] = (\sigma + 1) I_{\lambda,1}^m (a,c,A) f(z) - \sigma I_{\lambda,1}^m (a,c,A) L_\sigma f(z). \quad (56)$$

In order to obtain the integral-preserving properties involving the integral operator $L_\sigma$, we need the following lemma which is known as Jack's Lemma (see [14]).

Lemma 2. [14]. Let $\varphi(z)$ be a non-constant function analytic in $U$ with $\varphi(0) = 0$. If $|\varphi(z)|$ attains its maximum value on the circle $|z| = r < 1$ at $z_0$, then

$$z_0 \varphi'(z_0) = \zeta \varphi(z_0),$$

where $\zeta$ is a real number and $\zeta \geq 1$.

Theorem 5. Let $\sigma > -\alpha$. If $f(z) \in S_{\alpha,1}^\infty (a,c,A; \alpha)$, then

$$L_\sigma f(z) \in S_{\alpha,1}^\infty (a,c,A; \alpha).$$

Proof. Suppose that $f(z) \in S_{\alpha,1}^\infty (a,c,A; \alpha)$ and let

$$\frac{z \left[ I_{\lambda,1}^m (a,c,A) L_\sigma f(z) \right]}{I_{\lambda,1}^m (a,c,A) L_\sigma f(z)} = \frac{1 + (1 - \alpha) \varphi(z)}{1 - \varphi(z)}, \quad (57)$$

where $\varphi(0) = 0$. Then, by using (56) and (57), we have

$$\frac{I_{\lambda,1}^m (a,c,A) f(z)}{I_{\lambda,1}^m (a,c,A) L_\sigma f(z)} = \frac{\sigma + 1 + (1 - \sigma - 2 \alpha) \varphi(z)}{(\sigma + 1)(1 - \varphi(z))}, \quad (58)$$

which, upon logarithmic differentiation, yields

$$\frac{z \left[ I_{\lambda,1}^m (a,c,A) f(z) \right]}{I_{\lambda,1}^m (a,c,A) f(z)} = \frac{1 + (1 - \alpha) \varphi(z)}{1 - \varphi(z)} + \frac{z \varphi'(z)}{1 - \varphi(z)} + \frac{(1 - \sigma - 2 \alpha) z \varphi'(z)}{(1 - \sigma - 2 \alpha) \varphi(z) + \sigma + 1}, \quad (59)$$
so that
\[
\frac{z \left( I_{\lambda,l}^m (a,c,A)f(z) \right)}{I_{\lambda,l}^m (a,c,A)f(z)} - \alpha = \left( 1 - \alpha \right) \frac{1 + \omega(z)}{1 - \omega(z)} + \frac{z \omega'(z)}{1 - \omega(z)} + \frac{(1 - \sigma - 2\alpha)z \omega'(z)}{(1 - \sigma - 2\alpha)\omega(z) + \sigma + 1}. \tag{60}
\]

Now, assuming that \( \max_{|z|=1} |\omega(z)| = |\omega(z_0)| = 1 \) \( (z \in U) \) and applying Jack's lemma, we obtain
\[
z_0 \omega'(z_0) = \zeta \omega(z_0) \quad (\zeta \in \mathbb{R}, \zeta \geq 1).
\tag{61}
\]
If we set \( \omega(z_0) = e^{i\theta} \) \( (\theta \in \mathbb{R}) \) in (60) and observe that \[
\text{Re} \left\{ (1 - \alpha) \frac{1 + \omega(z_0)}{1 - \omega(z_0)} \right\} = 0,
\]
then, we obtain
\[
\text{Re} \left\{ z \left( I_{\lambda,l}^m (a,c,A)f(z) \right) - \alpha \right\} = \text{Re} \left\{ \frac{z_0 \omega'(z_0)}{1 - \omega(z_0)} + \frac{(1 - \sigma - 2\alpha)z_0 \omega'(z_0)}{(1 - \sigma - 2\alpha)\omega(z_0) + \sigma + 1} \right\}
\]
\[
= \text{Re} \left\{ \frac{\zeta e^{i\theta}}{1 - e^{i\theta}} + \frac{(1 - \sigma - 2\alpha)\zeta e^{i\theta}}{(1 - \sigma - 2\alpha)e^{i\theta} + \sigma + 1} \right\}
\]
\[
= -\frac{2\zeta (\sigma + \alpha)(1 - \alpha)}{2(1 + \sigma)(1 - \sigma - 2\alpha)\cos \theta + (1 + \sigma)^2 + (1 - \sigma - 2\alpha)^2}
\]
\< 0,
\]
which obviously contradicts the hypothesis \( f(z) \in S_{\lambda,l}^m (a,c,A;\alpha) \). Consequently, we can deduce that \(|\omega(z)| < 1 \) \( (z \in U) \), which, in view of (57), proves the integral-preserving property asserted by Theorem 5.

Taking \( a = c \) in Theorem 5, we obtain the following corollary.

**Corollary 3.** Let \( \sigma > -\alpha \). If \( f(z) \in S_{\lambda,l}^m (\alpha) \), then \( L_{a,\lambda} f(z) \in S_{\lambda,l}^m (\alpha) \), where the subclass \( S_{\lambda,l}^m (\alpha) \) is defined by (22).

**Remark 6.**
(i) Taking \( \lambda = 1, m = \mu > 0 \) and \( l = a - 1(a > 0) \) in Corollary 3, we obtain the results obtained by Aouf [1, Theorem 5];
(ii) Taking \( \lambda = l = 1 \) and \( m = \sigma(\sigma > 0) \) in Corollary 3, we obtain the results obtained by Liu [19, Theorem 5].

Taking \( m = 0 \) in Theorem 5, we obtain the following corollary.

**Corollary 4.** Let \( \sigma > -\alpha \). If \( f(z) \in S^*(a,c,A;\alpha) \), then \( L_{a,\lambda} f(z) \in S^*(a,c,A;\alpha) \), where the subclass \( S^*(a,c,A;\alpha) \) is defined by (26).

**Remark 7.** Taking \( \alpha = 0 \), \( a = \mu - 1(\mu > 0) \), \( c = \lambda(\lambda > -1) \) and \( A = 1 \) in Corollary 4, we obtain the results obtained by Choi et al. [9, Theorem 4, with \( \varphi(z) = \frac{1 + z}{1 - z} \)].
Theorem 6. Let $\sigma > -\alpha$. If $f(z) \in C_{\lambda,l}^n(a,c;A;\alpha)$, then

$$L_\sigma f(z) \in C_{\lambda,l}^n(a,c;A;\alpha).$$

Proof. By applying Theorem 5, it follows that

$$f(z) \in C_{\lambda,l}^m(a,c;A;\alpha) \Leftrightarrow zf'(z) \in S_{\lambda,l}^m(a,c;A;\alpha)$$

$$\Rightarrow L_\sigma (zf'(z)) \in S_{\lambda,l}^m(a,c;A;\alpha)$$

$$\Leftrightarrow z(L_\sigma f(z))' \in S_{\lambda,l}^m(a,c;A;\alpha)$$

$$\Leftrightarrow L_\sigma f(z) \in C_{\lambda,l}^m(a,c;A;\alpha),$$

which proves Theorem 6.

Taking $a = c$ in Theorem 6, we obtain the following corollary.

Corollary 5. Let $\sigma > -\alpha$. If $f(z) \in C_{\lambda,l}^m(\alpha)$, then $L_\sigma f(z) \in C_{\lambda,l}^m(\alpha)$, where the subclass $C_{\lambda,l}^m(\alpha)$ is defined by (23).

Remark 8.
(i) Taking $\lambda = 1$, $m = \mu(\mu > 0)$ and $l = a - 1(a > 0)$ in Corollary 5, we obtain the result obtained by Aouf [1, Theorem 6];
(ii) Taking $\lambda = l = 1$ and $m = \sigma(\sigma > 0)$ in Corollary 5, we obtain the result obtained by Liu [19, Theorem 6].

Taking $m = 0$ in Theorem 6, we obtain the following corollary.

Corollary 6. Let $\sigma > -\alpha$. If $f(z) \in C(a,c;A;\alpha)$, then $L_\sigma f(z) \in C(a,c;A;\alpha)$, where the subclass $C(a,c;A;\alpha)$ is defined by (27).

Remark 9. Taking $\alpha = 0$, $a = \mu - 1(\mu > 0)$, $c = \lambda(\lambda > -1)$ and $A = 1$ in Corollary 6, we obtain the results obtained by Choi et al.[9, Theorem 5, with $\varphi(z) = \frac{1 + z}{1 - z}$].

Theorem 7. Let $\sigma > -\alpha$. If $f(z) \in K_{\lambda,l}^m(a,c;A;\beta,\alpha)$, then

$$L_\sigma f(z) \in K_{\lambda,l}^m(a,c;A;\beta,\alpha).$$

Proof. Suppose that $f(z) \in K_{\lambda,l}^m(a,c;A;\beta,\alpha)$. Then, there exists a function $g(z) \in S_{\lambda,l}^m(a,c;A;\alpha)$ such that

$$Re\left\{\frac{z(I_{\lambda,l}^m(a,c;A)f(z))'}{I_{\lambda,l}^m(a,c;A)g(z)}\right\} > \beta.$$

Thus, upon setting

$$\frac{z(I_{\lambda,l}^m(a,c;A)L_\sigma f(z))'}{I_{\lambda,l}^m(a,c;A)L_\sigma g(z)} - \beta = (1 - \beta)h(z),$$

(62)
where \( h(z) \) is given by (31), we find from (56) that
\[
\frac{z \left( I_{\lambda, l}^m(a, c, A) f(z) \right)'}{I_{\lambda, l}^m(a, c, A) g(z)} = \frac{I_{\lambda, l}^m(a, c, A) z f'(z)}{I_{\lambda, l}^m(a, c, A) g(z)} + \sigma \left( I_{\lambda, l}^m(a, c, A) L_\sigma f'(z) \right)
\]
\[
= \frac{z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f'(z) \right) + \sigma \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right)}{I_{\lambda, l}^m(a, c, A) L_\sigma g(z)}
\]
\[
= \frac{z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f'(z) \right) + \sigma \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right)}{I_{\lambda, l}^m(a, c, A) L_\sigma g(z)} + \sigma.
\]
(63)

Since \( g(z) \in S_{\lambda, l}^m(a, c, A; \alpha) \), we know from Theorem 5 that \( L_\sigma g(z) \in S_{\lambda, l}^m(a, c, A; \alpha) \). Then, we can set
\[
\frac{z \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right)'}{I_{\lambda, l}^m(a, c, A) L_\sigma g(z)} = (1 - \alpha) H(z),
\]
where \( Re \{ H(z) \} > 0 \). Then, substituting (62) and (64) into (63), we have
\[
\frac{z \left( I_{\lambda, l}^m(a, c, A) f(z) \right)'}{I_{\lambda, l}^m(a, c, A) g(z)} = \frac{z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f'(z) \right) + \sigma \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right)}{(1 - \alpha) H(z) + \alpha} + \sigma.
\]
(65)

Also, we find from (62) that
\[
z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f(z) \right) = \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right) \left[ (1 - \beta) h(z) + \beta \right].
\]
(66)

Differentiating both sides of (66) with respect to \( z \), we obtain
\[
z \left( z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f(z) \right) \right)' = (1 - \beta) z h'(z) \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right) + z \left( I_{\lambda, l}^m(a, c, A) L_\sigma g(z) \right) \left[ (1 - \beta) h(z) + \beta \right],
\]
(67)

that is,
\[
z \left( z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f(z) \right) \right)' \frac{z \left( I_{\lambda, l}^m(a, c, A) L_\sigma f(z) \right)'}{I_{\lambda, l}^m(a, c, A) L_\sigma g(z)} = (1 - \beta) z h'(z) + \left[ (1 - \beta) h(z) + \beta \right][1 - \alpha] H(z) + \alpha.
\]
(68)

Substituting (68) into (65), we find that
\[
\frac{z \left( I_{\lambda, l}^m(a, c, A) f(z) \right)'}{I_{\lambda, l}^m(a, c, A) g(z)} - \beta = (1 - \beta) h(z) + \frac{z h'(z)}{(1 - \alpha) H(z) + \alpha} + \sigma.
\]
(69)

Then, by setting \( u = h(z) = u_1 + iu_2 \) and \( v = z h'(z) = v_1 + iv_2 \), we can define the function \( \psi(u, v) \) by
\[
\psi(u, v) = (1 - \beta) u + \frac{(1 - \beta) v}{[(1 - \alpha) H(z) + \alpha] + \sigma},
\]
where \( (u,v) \in D = \mathbb{C} \times \mathbb{C} \). The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved.
Remark 10.
(i) Taking \( m = s \ (s \in \mathbb{R}) \), \( \lambda = 1 \), \( a = \mu - 1(\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 5-7, we obtain the results obtained by Cho and Kim [8, Theorems 3.1-3.3 with \( \phi(z) = \psi(z) = \frac{1 + z}{1 - z} \)];
(ii) Taking \( a = \mu - 1(\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 5-7, we obtain the results obtained by Aouf and El-Ashwah [3, Theorems 4-6 with \( \phi(z) = \psi(z) = \frac{1 + z}{1 - z} \)];
(iii) Taking \( l = 0 \), \( a = \mu - 1(\mu > 0) \), \( c = 0 \) and \( A = 1 \) in Theorems 5-7, we obtain the results obtained by Aouf et al. [2, Theorems 4-6 with \( \phi(z) = \psi(z) = \frac{1 + z}{1 - z} \)].

Taking \( a = c \) in Theorem 7, we obtain the following corollary.

Corollary 7. Let \( \sigma > - \alpha \). If \( f(z) \in K_{\lambda,1}^m(\beta,\alpha) \), then \( L_\sigma f(z) \in K_{\lambda,1}^m(\beta,\alpha) \), where the subclass \( K_{\lambda,1}^m(\beta,\alpha) \) is defined by (24).

Remark 11.
(i) Taking \( \lambda = 1 \), \( m = \mu(\mu > 0) \) and \( l = a - 1(a > 0) \) in Corollary 7, we obtain the results obtained by Aouf [1, Theorem 7];
(ii) Taking \( \lambda = l = 1 \) and \( m = \sigma(\sigma > 0) \) in Corollary 7, we obtain the results obtained by Liu [19, Theorem 7].

Taking \( m = 0 \) in Theorem 7, we obtain the following corollary.

Corollary 8. Let \( \sigma > - \alpha \). If \( f(z) \in K(a,c,A;\beta,\alpha) \), then \( L_\sigma f(z) \in K(a,c,A;\beta,\alpha) \), where the subclass \( K(a,c,A;\beta,\alpha) \) is defined by (28).

Remark 12. Taking \( \alpha = \beta = 0 \), \( a = \mu - 1(\mu > 0) \), \( c = \lambda(\lambda > -1) \) and \( A = 1 \) in Corollary 8, we obtain the results obtained by Choi et al. [9, Theorem 6, with \( \phi(z) = \psi(z) = \frac{1 + z}{1 - z} \)].

Theorem 8. Let \( \sigma > - \alpha \). If \( f(z) \in K_{\lambda,1}^m(a,c,A;\beta,\alpha) \), then
\[ L_\sigma f(z) \in K_{\lambda,1}^m(a,c,A;\beta,\alpha). \]

Proof. Just as we derived Theorem 6 from Theorem 5. Easily, we can deduce Theorem 8 from Theorem 7. So we choose to omit the proof.

Taking \( a = c \) in Theorem 8, we obtain the following corollary.

Corollary 9. Let \( \sigma > - \alpha \). If \( f(z) \in K_{\lambda,1}^m(\beta,\alpha) \), then \( L_\sigma f(z) \in K_{\lambda,1}^m(\beta,\alpha) \), where the subclass \( K_{\lambda,1}^m(\beta,\alpha) \) is defined by (25).

Remark 13.
(i) Taking \( \lambda = 1 \), \( m = \mu(\mu > 0) \) and \( l = a - 1(a > 0) \) in Corollary 9, we obtain the results obtained by Aouf [1, Theorem 8];
(ii) Taking $\lambda = l = 1$ and $m = \sigma (\sigma > 0)$ in Corollary 9, we obtain the results obtained by Liu [19, Theorem 8].

Taking $m = 0$ in Theorem 8, we obtain the following corollary.

**Corollary 10.** Let $\sigma > -\alpha$. If $f(z) \in K^+(a,c,A;\beta,\alpha)$, then $L_{\sigma} f(z) \in K^+(a,c,A;\beta,\alpha)$, where the subclass $K^+(a,c,A;\beta,\alpha)$ is defined by (29).

### 4. CONCLUSION

From the results obtained in this paper, we can easily obtain the corresponding results of the well-known operators such as *Jung-Kim-Srivastava* operator, *Carlson-Shaffer* operator, *Noor* operator, *Ruscheweyeh* operator, *Salagean* operator, and others as a special choices of the parameters as mentioned before.

### 5. REFERENCES