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Research Article

The Study of Accuracy and Efficiency of ODE Solvers While Performing Numerical Simulations of Terrestrial Planets

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Abstract: The N-body problem is one of the well-known and most central computational problem. The N-body problem of the Solar System is not only a rich source of initial value problems (IVPs) for ordinary differential equations (ODEs), but is also very convenient to understand the orbital evolution of the Solar System; see, for example,[1, 2]. Wide range of numerical integrators have been developed and implemented for performing such N-body simulations. The main objective of this research paper is to analyze and compare the accuracy and efficiency of different ordinary differential equation (ODE) solvers applied to the Kepler's two-body problem for Terrestrial planets. Throughout this paper, the error growth is investigated in terms of global error in position and velocity, and the relative error in terms of angular momentum and total energy of the system. To quantify the quality of different ODE solvers, we performed numerical tests applied to the Kepler's two body problem for Terrestrial planets with local error tolerances ranging from 10^{-12} to 10^{-4}.

Keywords: Kepler's two body problem, N-body simulations, Terrestrial planets, ODE solvers

1. INTRODUCTION

In the dynamical astronomy and celestial mechanics, the numerical integration plays a vital role to investigate complex problems, for example, N-body problems. Computational astronomers make extensive use of accurate N-body simulations to investigate the orbital evolution of the planets, comets, asteroids, and other small celestial bodies in the Solar System. The numerical simulations of N-body problems are performed by first obtaining a set of second order ODEs for the acceleration of the N-bodies, and describing the positions and velocities of the N-bodies at the initial time $= t_0$. The initial value problem (IVP) we are considering is represented by:

$$y''(t) = f\big(t, y(t)\big),\ y(t_0) = y_0,\ y'(t_0) = y_0', \tag{1}$$

where, $y_0, y_0' \in R^k$ represent the initial positions and velocities, operator $'$ represents differentiation with respect to time, $f : R \times R^k \to R^k$ a smooth function, and k is the dimension of the above IVP. In the N-body problem, when N = 2 then the problem is in its simplest form, i.e., two-body problem. Large number of numerical integrators are developed and used to find numerical approximations of these types of problems; see, for example, Runge-Kutta [3, 4], ODE solvers for non-stiff problems [5, 6, 7].

2. MATERIALS AND METHODS

In celestial mechanics, the Kepler's two-body problem [10, 11] involves the motion of one body about another under the influence of their mutual gravitational attraction. The two-body problem is considered as the simplest problem in the dynamical astronomy, because the exact solution of the two-body problem

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exists. The best direct orbit calculations arise when the central body is largely heavier than the orbiting body, for example, the man-made satellite around the Earth and planetary orbits around the Sun. In this research work, we used Kepler's two-body problem for Terrestrial planets as the test problem. Terrestrial planets (inner planets), i.e., Mercury, Venus, Earth, and Mars are closest to the Sun [12, 13, 14]. The equations of motion of Kepler's two body problem are:

$$y_1'' = -\frac{y_1}{r^3}, \tag{2}$$

$$y_2'' = -\frac{y_2}{r^3}, \tag{3}$$

where, y_1,y_2 represent the $x-$ and $y-$components of one body corresponding to the other body, and $r = \sqrt{y_1^2 + y_2^2}$. The initial conditions are

$$y_1(0) = 1 - e, y_2(0) = 0, \quad y_1'(0) = 0, y_2'(0) = \sqrt{\frac{1+e}{1-e}}.$$

The parameter e is the orbital eccentricity and $0 \leq e < 1$. Since, the true solution of Kepler problem is available. Therefore, the Kepler's two body problem is quite useful for observing the accuracy of different ODE solvers over a short duration of time. The analytical solution of the Kepler's two body problem is given by

$$y_1 = \cos(\eta) - e, \qquad y_2 = \sqrt{1-e^2}\sin(\eta),$$

$$y_1' = -\sin(\eta)\left(1 - ecos(\eta)\right)^{-1}, \qquad y_2' = \sqrt{(1-e^2)}\cos(\eta)(1 - ecos(\eta))^{-1},$$

where, the eccentric anomaly η satisfies Kepler's equation $x = \eta - \mathrm{e}\sin(\eta)$.

Throughout this paper, we have discussed different types of errors. To quantify the quality of numerical approximations obtained by different ODE solvers, the global error is of main concern. The main source of error, for the total error in the system, is the integration error. The integration error consists of two types of errors, namely, truncation and round-off error. Since, computer stores number to a certain arithematic precision. Therefore, for accurate N-body simulations, round-off error can contribute significantly to the global error.

Suppose, $y_{num}(t)$ and $y_{true}(t)$ are position vectors of the solutions obtained numerically and analytically, respectively, and $y_{num}'(t)$ and $y_{true}'(t)$ are velocity vectors of numerical and analytical solutions, respectively. The L_2 -norm of global errors in the position and velocity are defined as

$E_r(t) = \|y_{num}(t) - y_{true}(t)\|_2$, $E_v(t) = \|y_{num}'(t) - y_{true}'(t)\|_2$, where, $\|.\|_2$ is the L_2 -norm.

Physical systems usually have conserved quantities, for example, angular momentum $L(t)$ and total energy $H(t)$. Generally, $L(t)$ and $H(t)$ are not conserved accurately by numerical approximations. However, this digression provides evaluation to quantify the quality of numerical approximations. The total energy $H(t)$ of the two-body system is defined as

$$H(t) = \frac{y_1'^2 + y_2'^2}{2} - \frac{1}{\sqrt{y_1^2 + y_2^2}}.$$

The relative error in energy is defined as

$$H_{rel}(t) = \left|\frac{H_0 - H(t)}{H_0}\right|,$$

where, H_0 is the total energy at $t = t_0$.

The total angular momentum $L(t)$ is defined as

$$L(t) = y_1 y_2' - y_2 y_1'.$$

The relative error in angular momentum is defined as

$$L_{rel}(t) = \frac{\|L_0 - L(t)\|_2}{\|L_0\|_2},$$

where, L_0 is the angular momentum at $t = t_0$. Notice that, unlike the global error in position and velocity, the exact solution is not required to calculate $H_{rel}(t)$ and $L_{rel}(t)$. Hence, fewer computing resources are required to observe the quality of different ODE solvers. Since, $H_{rel}$and L_{rel} are scalar quantities. So, in order to obtain small error, $H_{rel}$and L_{rel} require only one constraint. Whereas, for E_r and E_v, being vector quantities, each coordinate of E_r and E_v has to be small.

3. ODE SOLVERS

A wide range of numerical integration techniques have been developed for the numerical approximations of ODEs that correspond to the continuous state of dynamic systems. MATLAB is a software that is used to solve an extensive variety of such problems. The MATLAB ODE suit is a set of codes for solving first order systems of ODEs for IVPs and plotting mathematical results of these problems [5]. The ODE solvers control the estimated local error for initial value problems. A local error tolerance is specified and, if the estimated local error is too large comparative to this specific tolerance. Then the time-step is rejected and a new attempt is made with smaller time-step. All ODE solvers in MATLAB use the same function interface, so it is very easy to try several methods on the same problem and observe their behavior [8]. The most important non-stiff solvers are ODE23, ODE45, and ODE113 [5].

3.1 ODE45 Solver

The ODE45 solver is a popular (4, 5) embedded pair of Dormand and Prince [7]. The ODE45 solver consists of six-stage embedded pair of Runge-Kutta methods of order 4 and 5. The ODE45 solver is a very attractive one step solver for the numerical approximations of non-stiff problems. The ODE45 advances the solution with 5th order method and the local error is controlled by taking the difference between the numerical approximations obtained by 5th order and 4th order methods. In order to compute $y(t_n)$, ODE45 solver requires only the solution $y(t_{n-1})$ at the immediately preceding time-step. Generally, the ODE45 solver is the best solver to implement as a “first choice” for most of the problems.

3.2 ODE23 Solver

The ODE23 solver is based upon explicit Runge-Kutta (2, 3) embedded pair of Bogacki and Shampine [6]. In the presence of mild stiffness and at crude tolerances, the ODE23 solver may be more efficient than ODE45 solver. The ODE23 solver is a one step solver which is frequently used for non-stiff problems. The ODE23 solver consists of four-stage embedded pair of explicit Runge-Kutta methods of order 2 and 3. The ODE23 advances the solution with 3rd order method and the local error is controlled by taking the difference between the numerical approximations obtained by 3rd order and 2nd order methods.

3.3 ODE113 Solver

The ODE113 solver is a variable order and variable time-step solver. The ODE113 solver uses Adams-Bashforth-Moulton predictor-corrector methods of order 1 to 13 [9]. When the ODE function is very expensive to evaluate then at stringent tolerances, the ODE113 solver may be more efficient than ODE45 solver. Normally, the ODE113 solver requires solution at several preceding time-steps to obtain the current solution values [5].

4. RESULTS AND DISCUSSION

In this section, we investigate the global error in position and velocity, and the relative error in angular momentum and energy for Terrestrial planets. We perform numerical experiments for different numerical ODE solvers applied to the Kepler'stwo body problem for Terrestrial planets over the orbital time period of each of the planet with $TOL = [10^{-12} - 10^{-4}]$. The eccentricity of Mercury, Venus, Earth, and Mars is approximately 0.21, 0.0068, 0.017, and 0.093, respectively. Whereas, for the Terrestrial planets, the corresponding time of the orbital motion to complete one vibration is approximately 28π, 72π, 117π, and 219π, respectively. Different time-steps are chosen to perform these numerical experiments. We evaluate the position and velocity on each of the time-step using ODE solvers. The values of positions, velocities, and time are stored in separate files. Then we obtain error in positions and velocities with respect to the true solutions obtained at the saved values of time.

Fig. 1, shows four sets of experiments with time-steps π/2, π/4, π/8, and π/16 to obtain the maximum global error in position using ODE23, ODE45, and ODE113 integrators applied to the Sun-Mercury system with $TOL = [10^{-12} - 10^{-4}]$. Fig. 1(a) shows four sets of numerical experiments with time step $\pi/2$. From Fig. 1(a), we observe that ODE45 gives the least error, which is approximately 1.9929×10^{-9} for the Sun-Mercury system at a combination of tolerance 10^{-12} and time-step $\pi/2$. The ODE23 integrator gives the 2nd least error, which is approximately $3.6778 \times 10^{-9}$and the 3rd least error, which is approximately 5.8392×10^{-9} is attained by ODE113 integrator at the same combination of tolerance and time-step. Fig. 1(a) depicts a clear pattern. When the tolerance is increased, the maximum global error in position is also increased.

Fig. 1(b) shows the same set of experiments with time-step $\pi/4$. We see that at tolerances less than 10^{-10} all three integrators lose their accuracy by approximately two orders of magnitude as compared to the global errors at time-step $\pi/2$. When the time-step is $\pi/8$ and tolerance 10^{-12}, as shown in Fig. 1(c), all the three integrators have achieved almost the same accuracy, which is approximately 0.0272. Fig. 1(d) shows the experiments performed with time-step $\pi/16$. We observed that all three integrators lose their accuracy at tolerances less than 10^{-6} and give us straight lines which shows that there is no change in error when we further reduce the tolerance.

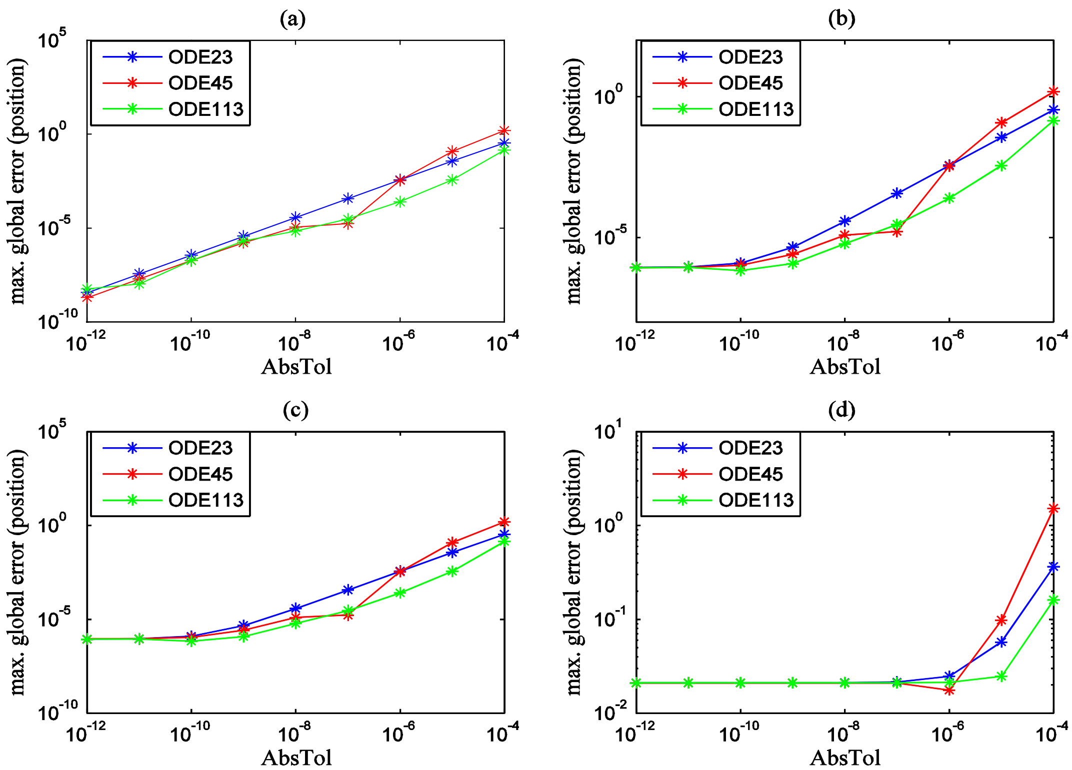

Fig. 1. The maximum global error in position using ODE23, ODE45, and ODE113 for the Sun-Mercury system against tolerances ranging from 10^{-12} to 10^{-4} and using time-steps from π/16 to π/2.

From all sets of experiments in Fig. 1, we observe that the accuracy of the given ODE solvers is improved if the time-step is increased at tolerance 10^{-12}. We conclude that a combination of tolerance $10^{-12}$and time-step π/2 gives better results in terms of maximum global error in position for all three integrators. We also observe that using ODE45 solver the least maximum global error in position is approximately 1.993×10^{-9}, which is the best observed accuracy. The ODE45 integrator achieved approximately 45% and 65% better accuracy than ODE23 and ODE113, respectively.

Fig. 2 shows four sets of experiments with time-steps π/2, π/4, π/8, and π/16 to obtain the maximum global error in position using ODE23, ODE45, and ODE113 integrators applied to the Sun-Venus system with $TOL = [10^{-12} - 10^{-4}]$. Fig. 2(a) shows four sets of numerical experiments with time step $\pi/2$. From Fig. 2(a), we observe that ODE113 gives the least error, which is approximately 5.0325×10^{-10} for the Sun-Venus system at a combination of tolerance 10^{-12} and time-step $\pi/2$. The ODE45 integrator gives the 2nd least error, which is approximately $1.0321\times 10^{-8}$and the 3rd least error, which is approximately 2.2642×10^{-8} is attained by ODE23 integrator at the same combination of tolerance and time-step. The ODE113 integrator achieved approximately 95% and 97% better accuracy than ODE45 and ODE23, respectively. Fig. 2(b) and Fig. 2(c), show the same sets of experiments with time-steps $\pi/4$ and $\pi/8$, respectively. We observed that at time-steps $\pi/4$ and $\pi/8$, the behavior of the global error obtained by using three integrators was very similar to that for the experiments performed with time-step $\pi/2$.

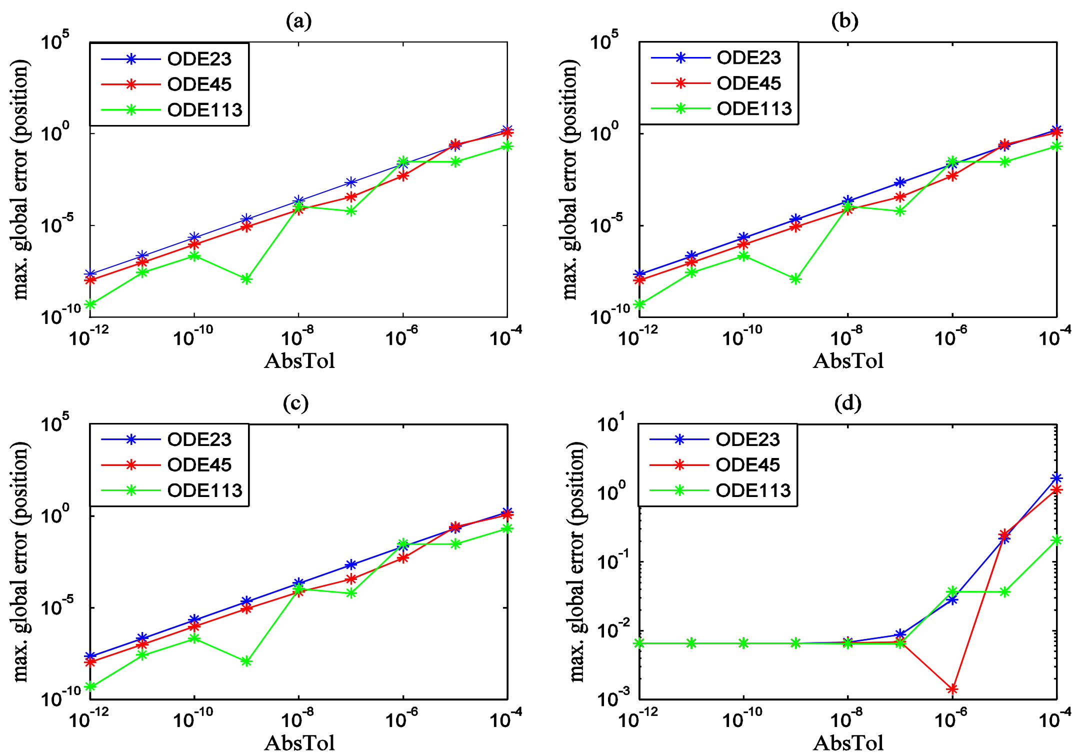

Fig. 2. The maximum global error in position using ODE23, ODE45, and ODE113 for the Sun-Venus system against tolerances ranging from 10^{-12} to 10^{-4} and using time-steps from π/16 to π/2.

Fig. 2(d) shows the experiments performed with time-step $\pi/16$. We observed that all three integrators lose their accuracy using tolerances less than 10^{-7} and give us a straight line which shows that there is no change in the error when we further reduce the tolerance. We observed from Fig. 2 that all three ODE solvers behave in a similar manner at time-steps $\pi/2$, $\pi/4$, and $\pi/8$ for the Sun-Venus system. However, for the efficiency reason, the time-step $\pi/2$ is recommended, because all the three integrators take least amount of CPU time with time-step $\pi/2$.

Fig. 3 shows four sets of experiments with time-steps π/2, π/4, π/8, and π/16 to obtain the maximum global error in position using ODE23, ODE45, and ODE113 integrators applied to the Sun-Earth system with $TOL = [10^{-12} - 10^{-4}]$. Fig. 3(a) shows four sets of numerical experiments with time step $\pi/2$. From Fig. 3(a) we observe that ODE23 gives the least error, which is approximately $6.3404\times 10^{-6}$for the

Sun-Earth system at a combination of tolerance 10^{-12} and time-step $\pi/2$. The ODE45 integrator gives the 2nd least error, which is approximately $6.3714\times 10^{-6}$and the 3rd least error, which is approximately 6.3974×10^{-6} is attained by ODE113 integrator at the same combination of tolerance and time-step. The ODE23 integrator achieved approximately 0.49% and 0.89% better accuracy than ODE45 and ODE113, respectively.

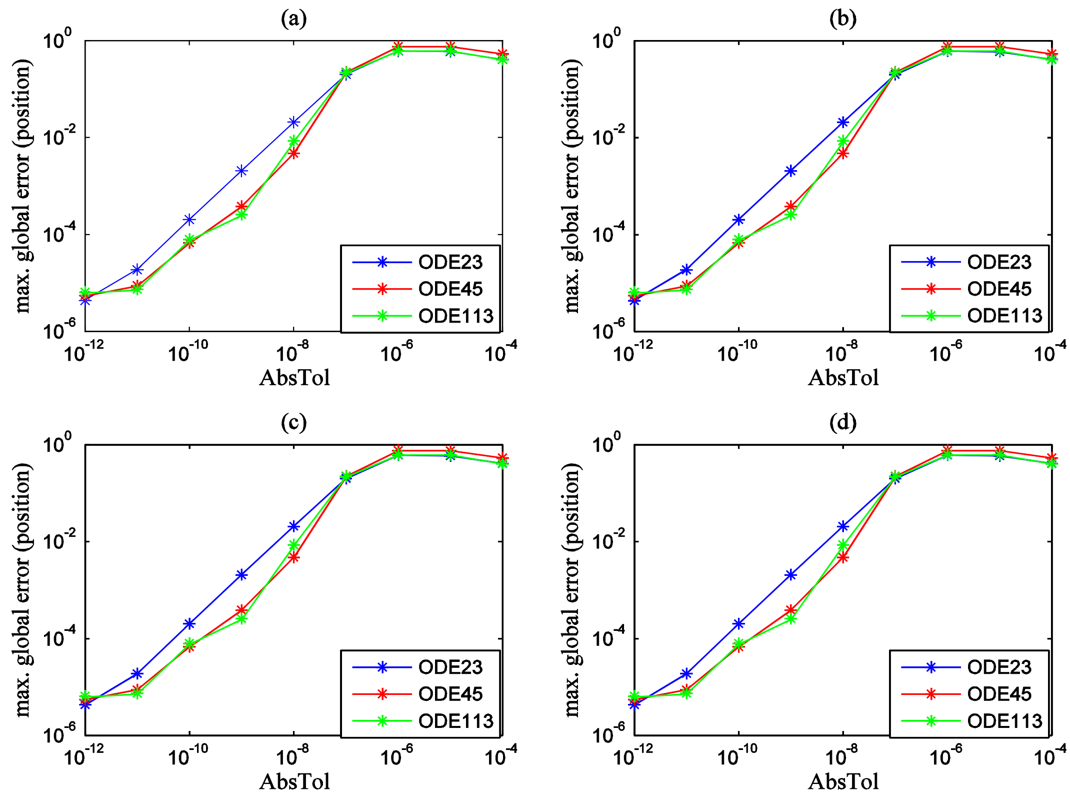

Fig. 3. The maximum global error in position using ODE23, ODE45, and ODE113 for the Sun-Earth system against tolerances ranging from 10^{-12} to 10^{-4} and using time-steps from π/16 to π/2.

Fig. 4 shows four sets of experiments with time-steps π/2, π/4, π/8, and π/16 to obtain the maximum global error in position using ODE23, ODE45, and ODE113 integrators applied to the Sun-Mars system with $TOL = [10^{-12} - 10^{-4}]$. Fig. 4(a) shows four sets of numerical experiments with time step $\pi/2$. From Fig. 4(a), we observe that ODE113 gives the least error, which is approximately 9.7285×10^{-8} for the Sun-Mars system at a combination of tolerance 10^{-12} and time-step $\pi/2$. The ODE45 integrator gives the 2nd least error, which is approximately $1.7289\times 10^{-7}$and the 3rd least error, which is approximately 2.6425×10^{-7} attained by ODE23 integrator at the same combination of tolerance and time-step. The ODE113 integrator achieved approximately 43% and 63% better accuracy than ODE45 and ODE23, respectively. When the time-step is $\pi/4$, as shown in Fig. 4(b), the behavior of the errors obtained by using three integrators is very much similar to that for the experiments performed with time-step $\pi/2$. Fig. 4(c) and Fig. 4(d) show that the experiments were performed with time-step $\pi/8$ and $\pi/16$, respectively. We observe that given integrators lose their accuracy using tolerances less than 10^{-7} and give us a straight line which shows that there is no change in error when we further reduce the tolerance. Furthermore, we have performed the previous sets of numerical experiments to investigate the error growth in velocity. We observed almost the same trend as of the errors in position but with certain orders of magnitude difference in accuracy.

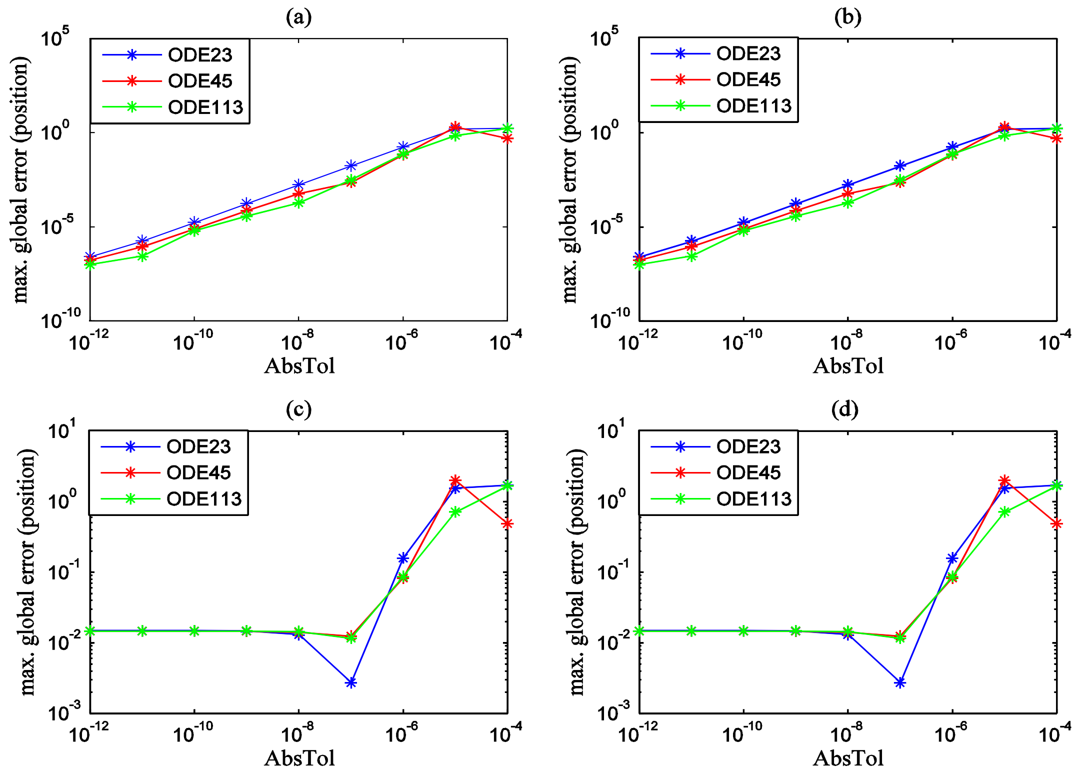

Fig. 4. The maximum global error in position using ODE23, ODE45, and ODE113 for the Sun-Mars system against tolerances ranging from 10^{-12} to 10^{-4} and using time-steps from $\pi/16$ to $\pi/2$.

Let us now consider the accuracy of the integrators ODE23, ODE45, and ODE113 using the combination of tolerance 10^{-12} and time-step $\pi/2$ over the orbital time period of each Terrestrial planet by estimating the relative error in energy and angular momentum.

Fig. 5 shows four sets of experiments with time-step of $\pi/2$ to observe the error behavior in total energy using three integrators ODE23, ODE45 and ODE113 applied to the Sun-Mercury, Sun-Venus, Sun-Earth, and Sun-Mars system over the orbital time period of each Terrestrial planet. The tolerance and time step is selected to give the smallest maximum global error. From Fig. 5(a), we observe that the best observed accuracy in terms of the relative error in energy is again achieved by the ODE45 integrator for the Sun-Mercury system. From Fig. 5(b), for the Sun-Venus system, we observe that the best observed accuracy in terms of relative error in energy is again achieved by the ODE113 integrator. From Fig. 5(c), we observe that the best observed accuracy in terms of relative error in energy is achieved by ODE113 rather than ODE23 for the Sun-Earth system. From Fig. 5(d), we observe for the Sun-Mars system that the best observed accuracy in terms of relative error in energy is again achieved by ODE113. As for Terretrial planets, we repeated the same sets of experiments and observe a similar behavior for the relative error in angular momentum.

Now we consider the efficiency of the ODE solvers discussed in this paper in terms of the amount of work done to attain a given accuracy. One way of observing the amount of work is to count the number of function evaluations against the maximum global error in position. Table 1 shows the number of function evaluations against the least maximum global error in position for the ODE45, ODE23 and ODE113 integrators with tolerance of 10^{-12} and time-step $\pi/2$.

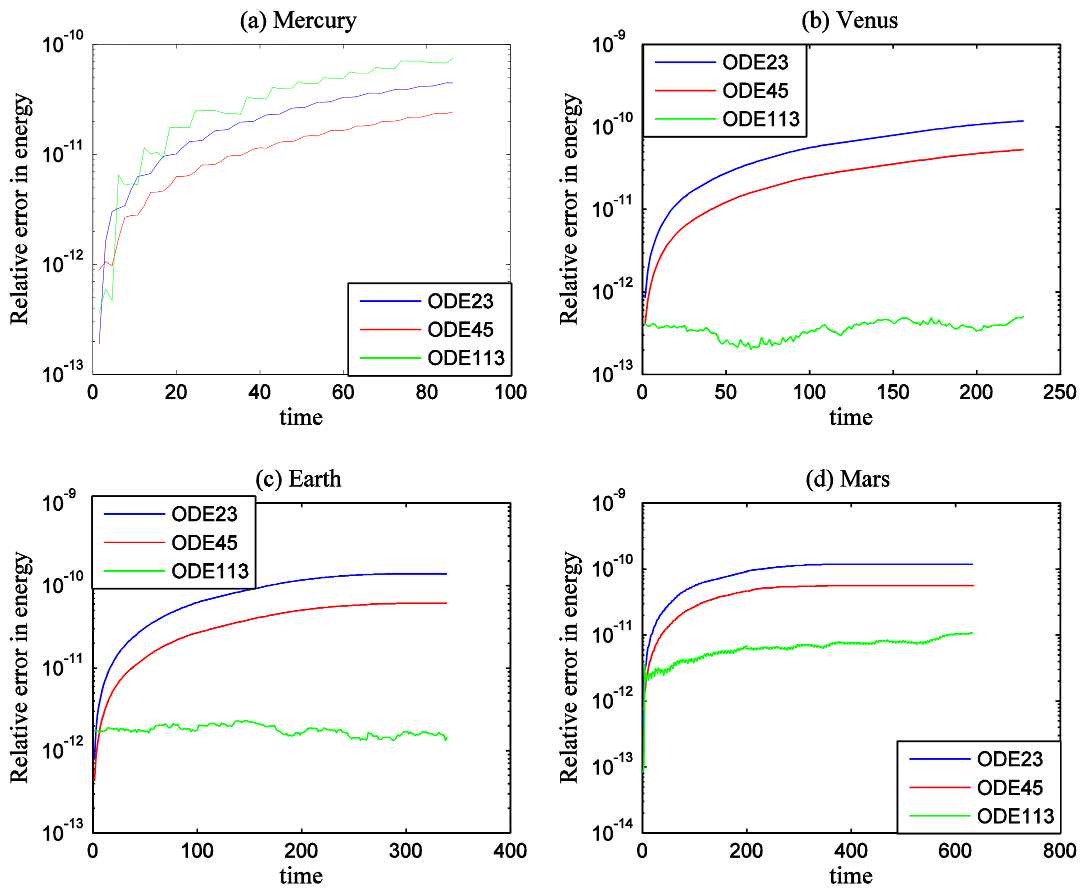

Fig. 5. The relative error in energy using ODE23, ODE45 and ODE113 integrators applied to the Sun-Mercury, Sun-Venus, Sun-Earth, and Sun-Mars system over the orbital time period of each Terrestrial planet.

For the Sun-Mercury system, the integrator ODE23 is approximately 25.4 times more expensive than ODE45 and the integrator ODE45 is approximately 9.13 times more expensive than ODE113. For the Sun-Venus system, the integrator ODE23 is approximately 22.5 times more expensive than ODE45 and the integrator ODE45 is approximately 14.2 times more expensive than ODE113 integrator. For the Sun-Earth system, the integrator ODE23 is approximately 22.6 times more expensive than ODE45 and the integrator ODE45 is approximately 13.9 times more expensive than ODE113. Whereas, the integrator ODE23 is approximately 23.6 times more expensive than ODE45 and the integrator ODE45 is approximately 11.4 times more expensive than ODE113 for the Sun-Mars system. Overall, we observe that the ODE113 integrator takes the least number of function evaluations for each Terrestrial planet and the ODE23 is the most expensive ODE solver, because it takes the most number of function evaluations.

Table 1. Number of function evaluations against least maximum global error in position for ODE23, ODE45 and ODE113 integrators at $h = \pi/2$ and tolerance $= 10^{-12}$over the orbital time period of each Terrestrial planet.

Solvers	Mercury	Venus	Earth	Mars
ODE23	995641	2258569	3688531	7180777
ODE45	39223	100375	163087	304303
ODE113	3966	7061	11679	26735

5. CONCLUSIONS

The main purpose of this paper was to observe the accuracy and efficiency of different ODE solvers applied to the real world problems involving the Sun and the Terrestrial planets. The simulations were performed over one orbital period of each of Terrestrial planet. For these simulations, we performed numerical experiments using three ODE integrators ODE23, ODE45, and ODE113 for the $TOL = [10^{-12} - 10^{-4}]$. We observed that when the tolerance is increased, the maximum global error in position is increased. We also observed that these ODE solvers are more accurate when the time-step is large at tolerance 10^{-12}. For the given range of tolerances from 10^{-12} to 10^{-4}, and time-steps from $\pi/16$ to $\pi/2$, we observed that for the Sun-Mercury system the integrator ODE45 achieves the best observed accuracy. The ODE45 integrator has achieved approximately 45% and 65% better accuracy than ODE23 and ODE113, respectively. For the Sun-Venus system, we observed that the integrator ODE113 achieves the best observed accuracy. The ODE113 integrator achieved approximately 95% and 97% better accuracy than ODE45 and ODE23, respectively. We observed that the integrator ODE23 achieves the best observed accuracy for the Sun-Earth system. The ODE23 integrator achieved approximately 0.49% and 0.89% better accuracy than ODE45 and ODE113, respectively. Finally we observed the results for the Sun-Mars system that the integrator ODE113 achieves the best observed accuracy. The ODE113 integrator achieved approximately 43% and 63% better accuracy than ODE45 and ODE23, respectively.

We also analyzed the efficiency of the ODE solvers discussed in this paper by counting the number of function evaluations against the least maximum global error in position for Terrestrial planets. We observed that the best observed accuracy attained by the integrator ODE45 for the Sun-Mercury system uses approximately 9.13 times more function evaluations than ODE113 and approximately 25.4 times less function evaluations than ODE23. The best observed accuracy attained by the integrator ODE113 for the Sun-Venus system uses approximately 14.2 times less function evaluations than ODE45 and approximately 319.9 times less function evaluations than ODE23. For the Sun-Earth system, the best observed accuracy attained by the integrator ODE23 uses approximately 22.6 times more function evaluations than ODE45 and approximately 315.8 times more function evaluations than ODE113. For the Sun-Mars system, the best accuracy attained by the integrator ODE113 uses approximately 11.5 times less function evaluations than ODE45 and approximately 268.6 times less function evaluations than ODE23.

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