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Research Article

Some Local-Value Relationships for the Recurrence Relation Related to the Tower of Hanoi Problem

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Abstract: Motivated by the recurrence relation satisfied by the 4-peg tower of the Hanoi Problem, Matsuura [1] has considered the generalized recurrence relation of the form

$$T(n, \alpha, \beta) = \min_{1 \le t \le n} \{ \alpha T(n-t, \alpha, \beta) + \beta S(t,3) \},$$

where α and β are natural numbers, and $S(t,3) = 2^t - 1$ is the solution of the 3-peg Tower of Hanoi problem with t discs. This paper studies more closely the above recurrence relation and gives some new relationships, including some local-value relationships. The Reve's puzzle is a particular case of the above recurrence relation with $\alpha = 2$.

Keywords : Tower of Hanoi, recurrence relation, local-value relationships

1. INTRODUCTION

The generalized 4-peg Tower of Hanoi problem (commonly known as the Reve`s puzzle), posed by Dudeney [2] is as follows : There are 4 pegs, designated as S, P_1, P_2, and D, and n discs D_1, D_2, …, D_n of different sizes, where D_1 is the smallest disc, D_2 is the second smallest, and so on with D_n being the largest. Initially, the discs rest on the source peg, S, in a tower in increasing order, with the largest disc D_n at the bottom, the second largest disc, D_{n-1}, above it, and so on, with the smallest disc D_1 at the top. The objective is to move the tower from the source peg S to the destination peg, D, in minimum number of moves, under the conditions that a move can transfer only the topmost disc from one peg to another such that no disc is ever placed on top of a smaller one.

Denoting by $M(n,4)$ the minimum number of moves required to solve the 4-peg Tower of Hanoi problem, the scheme followed is :

Step 1 : Move (optimally) the topmost n – k (smallest) discs from the source peg, S, to one of the auxiliary peg, say, P_1, using the 4 pegs, in $M(n-k, 4)$ moves,

Step 2 : Move (optimally) the tower of the largest k discs from the peg S to the destination peg, D, using the 3 pegs available, in M(k, 3) moves,

Step 3 : Move (optimally) the $n-k$ discs from the peg P_1 to the peg D, in $M(n-k, 4)$ moves,

where k is to be determined such that the total number of moves is minimum.

The above scheme leads to the following dynamic programming equation :

$$M(n, 4) = \min_{0 \le k \le n-1} \{ 2M(k, 4) + M(n-k, 3) \}, \tag{1.1}$$

$$M(0,4) = 0; \; M(n, 3) = 2^n - 1, \; n \ge 0. \tag{1.2}$$

Motivated by the recurrence relation (1.1), Matsuura [1] considers the following recurrence relation

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$$T(n,\alpha,\beta) = \min_{1 \le t \le n} \{ \alpha T(n-t,\alpha,\beta) + \beta S(t,3) \}, \quad (1.3)$$

$$T(0,\alpha,\beta) = 0;\ S(t,3) = 2^t - 1,\ t \ge 0, \quad (1.4)$$

where α and β are natural numbers. It may be mentioned here that, the particular case $T(n,2,1) = M(n,4)$.

This paper gives some local-value relationships involving $T(0,\alpha,\beta)$, in connection with the recurrence relation (1.3). They are given in Section 3. In Section 2, we give some preliminary results, that would be required to prove the results in Section 3. We conclude the paper with some remarks in Section 4.

2. SOME PRELIMINARY RESULTS

The result below has been established by Matsuura [1].

Lemma 2.1 : For any natural numbers α and β,

$$T(n,\alpha,\beta) = \beta T(n,\alpha,1),\ n \ge 1.$$

In view of the Lemma 2.1, it is sufficient to consider $T(n,\alpha,1)=T(n,\alpha)$, say, where $T(n,\alpha)$ satisfies the following recurrence relation :

$$T(n,\alpha) = \min_{1 \le t \le n} \{ \alpha T(n-t,\alpha) + S(t,3) \}, \quad (2.1)$$

$$T(0,\alpha) = 0. \quad (2.2)$$

An equivalent form of (2.1) is the following :

$$T(n,\alpha) = \min_{0 \le k \le n-1} \{ \alpha T(k,\alpha) + S(n-k,3) \}, \quad (2.3)$$

The following result deals with the special case $T(n,1)$.

Proposition 2.1 : For any $n \ge 1$, $T(n,1) = n$.

Proof : The proof is by induction on n. By (2.1), $T(1,1) = 1$, so that the result is true for n=1. So, we assume that the result is true for some n (so that $T(i,1)=iS(1,3)$ for all $1 \le i \le n$). Now, using the induction hypothesis, we get

$$T(n+1,1) = \min_{1 \le t \le n+1} \{ T(n+1-t,1) + S(t,3) \},$$

$$= \min_{1 \le t \le n+1} \{ (n+1-t)S(1,3) + S(t,3) \}.$$

Now, the sequence of numbers $\{(n+1-t) + S(t,3)\}_{t=1}^{n}$ is strictly increasing in t, since

$$(n+1-t-1) + S(t+1,3) > (n+1-t) + S(t,3)$$

if and only if

$$S(t+1,3) - S(t,3) > 1,$$

which is true for all $t \ge 1$. Thus, $T(n+1,1)$ is attained at $t=1$, so that

$$T(n+1,1) = n+1,$$

completing induction.

From Proposition 2.1, we see that, it is sufficient to consider the case when $\alpha \ge 2$. Since the case x =2 has been treated in more detail in Majumdar [3], it is, in fact, sufficient to concentrate our attention to the case when $\alpha \ge 3$.

For small values of n, the explicit forms of $T(n,\alpha)$ can be derived. This is done in the following Lemmas 2.2 – 2.7.

Lemma 2.2 : $T(2,\alpha)=3$ for all $\alpha \geq 2$.

Proof : By definition,

$$T(2,\alpha) = \min_{1\leq t\leq 2} \{\alpha T(2-t,\alpha)+S(t,3)\}$$
$$= \min\{\alpha T(1,\alpha)+S(1,3), S(2,3)\}$$
$$= \min\{\alpha+1, 3\},$$

from which the result follows immediately.

Lemma 2.3 : $T(3,\alpha)=\begin{cases}\alpha+3, \text{if } 2\leq\alpha\leq 4\\ 7, \quad \text{if } \alpha\geq 4\end{cases}$

Proof : Using the definition, together with Lemma 2.2, we get,

$$T(3,\alpha) = \min_{1\leq t\leq 3} \{\alpha T(3-t,\ \alpha)+S(t,\ 3)\}$$
$$= \min\{\alpha T(2,\alpha)+S(1,3), \alpha T(1,\alpha)+S(2,3), S(3,3)\}$$
$$= \min\{3\alpha+1, \alpha+3, 7\}\}.$$

Now, for any $\alpha\geq 2$, $3\alpha+1 > \alpha+3$; moreover,

$$\alpha+3 > 7 \text{ if and only if } \alpha\geq 4.$$

All these establish the lemma.

Corollary 2.1 : $T(3,\alpha)-T(2,\alpha)=\begin{cases}\alpha, \text{ if } 2\leq\alpha\leq 4\\ 4, \text{ if } \alpha\geq 4\end{cases}$

Lemma 2.4 : $T(4,\alpha)=\begin{cases}\alpha+7, \text{if } 2\leq\alpha\leq 8\\ 15, \quad \text{if } \alpha\geq 8\end{cases}$

Proof : By definition, together with Lemma 2.2 and Lemma 2.3, we have

$$T(4,\alpha) = \min_{1\leq t\leq 4} \{\alpha T(4-t,\alpha)+S(t,3)\}$$
$$= \min\{\alpha T(3,\alpha)+S(1,3), \alpha T(2,\alpha)+S(2,3), \alpha T(1,\alpha)+S(3,3), S(4,3)\}$$
$$= \min\{\alpha T(3,\alpha)+1, 3(\alpha+1), \alpha+7, 15\}\}.$$

Now, for any $\alpha\geq 2$, $3(\alpha+1) > \alpha+7$; and $\alpha+7 > 15$ if and only if $\alpha\geq 8$.

Also, for any $\alpha\geq 2$,

$$\alpha(\alpha+3)+1 > \alpha+7,\ 7\alpha+1 > \alpha+7.$$

Thus, we get the desired expression for $T(4,\alpha)$.

Corollary 2.2 : $T(4,\alpha)-T(3,\alpha)=\begin{cases}4, \text{ if } 2\leq\alpha\leq 4\\ \alpha, \text{ if } 4\leq\alpha\leq 16\\ 8, \text{ if } \alpha\geq 16\end{cases}$

Lemma 2.5 : $T(5,\alpha)=\begin{cases}3\alpha+7, \text{ if } 2\leq\alpha\leq 4\\ \alpha+15, \text{ if } 4\leq\alpha\leq 16\\ 31, \quad \text{if } \alpha\geq 16\end{cases}$

Proof : By definition, together with Lemma 2.2 – Lemma 2.4, we have

$$T(5,\alpha) = \min_{1 \le t \le 5} \{\alpha T(5-t,\alpha) + S(t,3)\}$$

$$= \min\{\alpha T(4,\alpha) + S(1,3), \alpha T(3,\alpha) + S(2,3), \alpha T(2,\alpha) + S(3,3),$$
$$\alpha T(1,\alpha) + S(4,3), S(5,3)\}$$

$$= \min\{\alpha T(4,\alpha) + 1, \alpha T(3,\alpha) + 3, 3\alpha + 7, \alpha + 15, 31\}$$

$$= \min\{3\alpha + 7, \alpha + 15, 31\},$$

where the last equality follows by virtue of Corollary 2.1 and Corollary 2.2.
Now,

$$3\alpha + 7 \ge \alpha + 15 \text{ if and only if } \alpha \ge 4,$$

and

$$S(5,3) = 31 \le \alpha + 15 \text{ if and only if } \alpha \ge 16.$$

All these complete the proof of the lemma.

Corollary 2.3 : $T(5,\alpha) - T(4,\alpha) = \begin{cases} 2\alpha, & \text{if } 2 \le \alpha \le 4 \\ 8, & \text{if } 4 \le \alpha \le 8 \\ \alpha, & \text{if } 8 \le \alpha \le 16 \\ 16, & \text{if } \alpha \ge 16 \end{cases}$

Lemma 2.6 : $T(6,\alpha) = \begin{cases} 3\alpha + 15, & \text{if } 3 \le \alpha \le 8 \\ \alpha + 31, & \text{if } 8 \le \alpha \le 32 \\ 31, & \text{if } \alpha \ge 32 \end{cases}$

Proof : Using Lemma 2.2 – Lemma 2.4, as well as Corollaries 2.1 – 2.3, we have

$$T(6,\alpha) = \min_{1 \le t \le 6} \{\alpha T(6-t,\alpha) + S(t,3)\}$$

$$= \min\{\alpha T(5,\alpha) + S(1,3), \alpha T(4,\alpha) + S(2,3), \alpha T(3,\alpha) + S(3,3),$$
$$\alpha T(2,\alpha) + S(4,3), \alpha T(1,\alpha) + S(5,3), S(6,3)\}$$

$$= \min\{\alpha T(5,\alpha) + 1, \alpha T(4,\alpha) + 3, \alpha T(3,\alpha) + 7, 3\alpha + 15, \alpha + 31, 63\}$$

$$= \min\{3\alpha + 15, \alpha + 31, 63\}.$$

Now,

$$3\alpha + 15 \ge \alpha + 31 \text{ if and only if } \alpha \ge 8,$$

and

$$\alpha + 31 \ge 63 \text{ if and only if } \alpha \ge 32.$$

Hence the lemma.

Corollary 2.4 : $T(6,\alpha) - T(5,\alpha) = \begin{cases} 8, & \text{if } 3 \le \alpha \le 4 \\ 2\alpha, & \text{if } 4 \le \alpha \le 8 \\ 16, & \text{if } 8 \le \alpha \le 16 \\ \alpha, & \text{if } 16 \le \alpha \le 32 \\ 32, & \text{if } \alpha \ge 32 \end{cases}$

Lemma 2.7 : $T(7,\alpha)=\begin{cases}\alpha(\alpha+3)+15, & \text{if } 2\le\alpha\le 4\\ 3\alpha+31, & \text{if } 4\le\alpha\le 16\\ \alpha+63, & \text{if } 16\le\alpha\le 64\\ 127, & \text{if } \alpha\ge 64\end{cases}$

Proof : Using Lemma 2.2 – Lemma 2.4 and Corollaries 2.1 – 2.3, we have

$$T(7,\alpha)=\min_{1\le t\le 7}\left\{\alpha T(7-t,\alpha)+S(t,3)\right\}$$

$$=\min\{\alpha T(6,\alpha)+S(1,3),\ \alpha T(5,\alpha)+S(2,3),\ \alpha T(4,\alpha)+S(3,3),$$
$$\alpha T(3,\alpha)+S(4,3),\ \alpha T(2,\alpha)+S(5,3),\ \alpha T(1,\alpha)+S(6,3),\ S(7,3)\}$$

$$=\min\{\alpha T(6,\alpha)+1,\ \alpha T(5,\alpha)+3,\ \alpha T(4,\alpha)+7,\ \alpha T(3,\alpha)+15,\ 3\alpha+31,\ \alpha+63,\ 127\}$$

$$=\min\{\alpha T(3,\alpha)+15,\ 3\alpha+31,\ \alpha+63,\ 127\}.$$

Now, if $2\le\alpha\le 4$,

$$\alpha(\alpha+3)+15\le 3\alpha+31,$$

and if $\alpha\ge 4$,

$$7\alpha+15\ge 3\alpha+31.$$

Also,

$$3\alpha+31\ge\alpha+63 \text{ if and only if } \alpha\ge 16,$$

and

$$\alpha+63\ge 127 \text{ if and only if } \alpha\ge 64.$$

Thus, we get the desired expression for $T(6,\alpha)$.

Corollary 2.5 : $T(7,\alpha)-T(6,\alpha)=\begin{cases}\alpha^2, & \text{if } 3\le\alpha\le 4\\ 16, & \text{if } 4\le\alpha\le 8\\ 2\alpha, & \text{if } 8\le\alpha\le 16\\ 32, & \text{if } 16\le\alpha\le 32\\ \alpha, & \text{if } 32\le\alpha\le 64\\ 64, & \text{if } \alpha\ge 64\end{cases}$

In Corollaries 2.1 – 2.5, we give the expressions for $T(i+1,\alpha)-T(i,\alpha)$ for $1\le i\le 6$. In determining the values of $T(n+1,\alpha)$, these differences are vital, since

$$T(n+1,\alpha)=\sum_{i=1}^{n+1}[T(i,\alpha)-T(i-1,\alpha)]$$

In the next Section 3, we derive some local-value relationships involving the functions $T(n,\alpha)$, $T(n+1,\alpha)$ and $T(n+2,\alpha)$.

3. SOME LOCAL-VALUE RELATIONSHIPS

We start with the following lemma.

Lemma 3.1 : For any $\alpha\ge 2$ fixed, $T(n,\alpha)$ is strictly increasing in n, that is,

$$T(n+1,\alpha)-T(n,\alpha)>0 \text{ for all } n\ (\ge 1).$$

Proof : Let $T(n+1,\alpha)$ be attained at $t=t_1$, that is, let

$$T(n+1,\alpha) = \alpha T(n+1 - t_1, \alpha) + S(t_1, 3).$$

We now consider the following two cases that may result :
Case (1) : When $t_1 = n+1$.
Here,

$$T(n+1,\alpha) = S(n+1,3) < \alpha T(1,\alpha) + S(n,3) = \alpha + S(n,3)$$

$$\Rightarrow \alpha > S(n+1,3) - S(n,3) = 2^n. \quad (1)$$

We now want to show that $T(n, \alpha)$ is attained at $t = n$, that is, $T(n,\alpha) = S(n,3)$. The proof is by contradiction. So, let $T(n, \alpha)$ be attained at some $t = t_2$ with $1 \le t_2 < n$. Then,

$$T(n,\alpha) = \alpha T(n - t_2, \alpha) + S(t_2, 3) < S(n,3)$$

$$\Rightarrow \alpha\, T(n - t_2, \alpha) < S(n,3) - S(t_2,3) < 2^n,$$

and we reach to a contradiction by virtue of (1). Therefore, $T(n, \alpha) = S(n,3)$, so that

$$T(n+1,\alpha) - T(n,\alpha) = S(n+1,3) - S(n,3) = 2^n > 0.$$

Case (2) : When $1 \le t_1 < n+1$.

In this case, the proof is by induction on n. Since

$$T(2,\alpha) = S(2,3) > T(1,\alpha) = S(1,3) \text{ for any } \alpha \ge 2,$$

we see that the result is true for n=1. So, we assume that the result is true for some n. Then, we need only prove that the result is true for n+1 as well. Now, since

$$T(n,\alpha) \le \alpha T(n - t_1,\alpha) + S(t_1,3),$$

it follows that

$$T(n+1,\alpha) - T(n,\alpha) \ge \alpha[T(n+1 - t_1,\alpha) - T(n - t_1,\alpha)] > 0,$$

where the last inequality follows by virtue of the induction hypothesis.
All these complete the proof of the lemma.

The corollary below follows immediately from Lemma 3.1, noting that

$$T(2,\alpha) - T(1,\alpha) = 2.$$

Corollary 3.1 : For any $\alpha \ge 2$ fixed, $T(n+1,\alpha) - T(n,\alpha) \ge 2$ for all $n\ (\ge 1)$.

In course of proving Lemma 3.1, we also proved the following results.

Corollary 3.2 : For any $\alpha \ge 2$ fixed, if $T(n+1, \alpha)$ is attained at $t = n+1$, then $T(n, \alpha)$ is attained at $t = n$.

Corollary 3.3 : For any $\alpha \ge 2$ fixed, if $T(n, \alpha)$ is attained at $t = n$, then $\alpha > 2^{n-1}$ $(n \ge 2)$.
Lemma 3.2 : For $n \ge 2$ and $\alpha \ge 3$, $T(n, \alpha)$ is not attained at $t = 1$.

Proof : We note that

$$\alpha T(n-1,\alpha) + S(1,3) > \alpha T(n-2,\alpha) + S(2,3)$$

if and only if

$$\alpha[T(n-1,\alpha) - T(n-2,\alpha)] > S(2,3) - S(1,3) = 2,$$

which is true (by Lemma 3.1) for any $n \ge 2$ and any $\alpha \ge 3$.

Lemma 3.3 : For any $\alpha \geq 3$ fixed, let $T(n, \alpha)$ be attained at $k = k_1$ and $T(n+1, \alpha)$ be attained at $k = k_2$. Then, $k_2 \geq k_1$.

Proof : Since (by (1.3)),

$$T(n, \alpha) = \alpha T(k_1, \alpha) + S(n - k_1, 3)$$
$$\leq \alpha T(k_2, \alpha) + S(n - k_2, 3),$$
$$T(n+1, \alpha) = \alpha T(k_2, \alpha) + S(n+1 - k_2, 3)$$
$$\leq \alpha T(k_1, \alpha) + S(n+1 - k_1, 3),$$

we get the following chain of inequalities :

$$2^{n-k_2} = S(n+1 - k_2, 3) - S(n - k_2, 3)$$
$$\leq T(n+1, \alpha) - T(n, \alpha)$$
$$\leq S(n+1 - k_1, 3) - S(n - k_1, 3) = 2^{n-k_1}. \tag{2}$$

Then, we must have $n - k_2 \leq n - k_1$, giving the result desired.

Corollary 3.4 : For any $\alpha \geq 3$ fixed, let $T(n, \alpha)$ be attained at $t = t_1$ and $T(n+1, \alpha)$ be attained at $t = t_2$. Then, $t_2 \leq t_1 + 1$.

Proof : follows immediately from Lemma 3.3, since $t_1 = n - k_1$, $t_2 = n + 1 - k_2$.

Lemma 3.4 : For any $\alpha \geq 3$ fixed,

(a) let $T(n, \alpha)$ be attained at $t = t_1$ and $T(n+1, \alpha)$ be attained at $t = t_2$; then, $t_2 \geq t_1$,

(b) for all $n \geq 1$,

$$T(n+1, \alpha) - T(n, \alpha) \leq T(n+2, \alpha) - T(n+1, \alpha)$$
$$\leq 2[T(n+1, \alpha) - T(n, \alpha)].$$

Proof : To prove part (b), we consider all the three possible cases that may arise.

Case (1) : When $T(n+2, \alpha) = S(n+2, 3)$.

In this case, by Corollary 3.2,

$$T(n+1, \alpha) = S(n+1, 3),\ T(n, \alpha) = S(n, 3),$$

so that

$$T(n+2, \alpha) - T(n+1, \alpha) = 2^{n+1} = 2[T(n+1, \alpha) - T(n, \alpha)].$$

Case (2) : When $T(n+1, \alpha) = S(n+1, 3)$.

Here, by Corollary 3.2, $T(n, \alpha) = S(n, 3)$, so that

$$T(n+1, \alpha) - T(n, \alpha) = 2^n.$$

Now, let $T(n+2, \alpha)$ be attained at $t = t_1$ for some $1 \leq t_1 \leq n+1$, that is, let

$$T(n+2, \alpha) = \alpha T(n+2 - t_1, \alpha) + S(t_1, 3) \leq S(n+2, 3).$$

Then, since

$$T(n+1, \alpha) \leq \alpha T(n+1 - t_1, \alpha) + S(t_1, 3),$$

we get the following chain of inequalities :

$$T(n+2, \alpha) - T(n+1, \alpha) \geq \alpha[T(n+2 - t_1, \alpha) - T(n+1 - t_1, \alpha)]$$
$$> \alpha > 2^n = T(n+1, \alpha) - T(n, \alpha)$$

where the last two inequalities follow by virtue of Corollary 3.1 and Corollary 3.3.
Also,

$$T(n+2,\alpha) - T(n+1,\alpha) \le 2^{n+1} = 2[T(n+1,\alpha) - T(n,\alpha)].$$

Case (3) : When $T(n+2,\alpha) \neq S(n+2,3)$ and $T(n+1,\alpha) \neq S(n+1,3)$.

This case and part (a) of the lemma is proved by induction on n. By Corollary 2.1,

$$T(3,\alpha) - T(2,\alpha) \ge 2 = T(2,\alpha) - T(1,\alpha) \text{ for any } \alpha \ge 3.$$

Thus, part (b) of the lemma holds true for $n = 1$. So, we assume the validity of the result for some n.

Now, let $T(n,\alpha)$ be attained at $t = t_1$ and $T(n+1,\alpha)$ be attained at $t = t_2$ with $t_1 > t_2$. Then,

$$T(n,\alpha) = \alpha T(n - t_1,\alpha) + S(t_1, 3) < \alpha T(n - t_2,\alpha) + S(t_2, 3),$$

and we get the following chain of inequalities :

$$\alpha[T(n+1 - t_2,\alpha) - T(n - t_2, \alpha)]$$
$$< T(n+1,\alpha) - T(n,\alpha) \le \alpha[T(n+1 - t_1,\alpha) - T(n - t_1, \alpha)],$$

which contradicts the induction hypothesis, since $n+1 - t_2 > n+1 - t_1$. Thus, $t_2 \ge t_1$, which we wanted to prove.

To complete the proof of part (b), let $T(n,\alpha)$ be attained at $k = k_1$, $T(n+1,\alpha)$ be attained at $k = k_2$ and $T(n+2,\alpha)$ be attained at $k = k_3$. By part (a) of the lemma and Lemma 3.3, we need to consider the following four cases :

Case (A) : When $k_1 = k_2 = k_3 = K$, say.

In this case,

$$T(n+2, \alpha) - T(n+1, \alpha) = 2^{n-K+1} = 2[T(n+1, \alpha) - T(n, \alpha)].$$

Case (B) : When $k_1 = k_2 = K, k_3 = K + 1$.
Here,

$$T(n+2, \alpha) - T(n+1, \alpha) > 2^{n-K} = T(n+1, \alpha) - T(n, \alpha).$$

Also,

$$T(n+2, \alpha) - T(n+1, \alpha) < 2^{n+1-K} = 2[T(n+1, \alpha) - T(n, \alpha)].$$

Case (C) : When $k_1 = K, k_2 = k_3 = K + 1$.

In this case,

$$T(n+2, \alpha) - T(n+1, \alpha) = 2^{n-K} \ge T(n+1, \alpha) - T(n, \alpha).$$

Again, since

$$T(n+1, \alpha) - T(n, \alpha) \ge 2^{n-K-1},$$

we see that

$$T(n+2, \alpha) - T(n+1, \alpha) \le 2[T(n+1, \alpha) - T(n, \alpha)].$$

Case (D) : When $k_1 = K, k_2 = K + 1, k_3 = K + 2$.

Here,

$$T(n+2, \alpha) - T(n+1, \alpha) = \alpha[T(K+2, \alpha) - T(K+1, \alpha)],$$
$$T(n+1, \alpha) - T(n, \alpha) = \alpha[T(K+1, \alpha) - T(K, \alpha)],$$

so that the result follows by virtue of the induction hypothesis.
All these complete the proof of the lemma.

From part (a) of Lemma 3.4 together with Corollary 3.1, we see that, if $T(n, \alpha)$ isattained at $t = t_1$, and $T(n+1, \alpha)$ is attained at $t = t_2$, then $t_1 \le t_2 \le t_1 + 1$. From the computational point of view, this allows to calculate recursively the value(s) of t where $T(n+1, \alpha)$ is attained, starting with the value(s) of t at which $T(n, \alpha)$ is attained. Part (b) of Lemma 3.4 shows that, for any $\alpha \ge 3$ fixed, $T(n, \alpha)$ is convex in n in the sense of the inequality. It also shows that for any $\alpha \ge 3$ fixed, $T(n+1, \alpha) - T(n, \alpha)$ is increasing (non-decreasing) in n.

Lemma 3.5 : For some $\alpha \ge 3$ and $\alpha \ge 3$, let $T(n, \alpha)$ be attained at the values $k = k_1$ and $k = k_2$. Then, $T(N, \alpha)$ is attained at all $k_1 \le k \le k_2$.

Proof : Let $T(n, \alpha)$ be attained at the values $k = k_1$ and $k = k_2$, so that

$$\alpha[T(k_2, \alpha) - T(k_1, \alpha)] = 2^{n-k_1} - 2^{n-k_2}.$$

There is nothing to prove if $k_2 = k_1 + 1$. So, let $k_2 \ge k_1 + 2$.
The proof is by contradiction. So, we assume that $T(n, \alpha)$ is not attained at $k = k_1 + 1$, so that

$$\alpha[T(k_1 + 1, \alpha) - T(k_1, \alpha)] > 2^{n-k_1-1}.$$

Let $k_2 = k_1 + m$ for some integer $m \ge 2$. Then, by part (b) of Lemma 3.4, together with the above inequality, we get the following chain of inequalities :

$$\begin{aligned} 2^{n-k_1} - 2^{n-k_2} &= \alpha[T(k_2, \alpha) - T(k_1, \alpha)] \\ &= \alpha[T(k_1 + m, \alpha) - T(k_1, \alpha)] \\ &= \alpha \sum_{i=1}^{m} [T(k_1 + i, \alpha) - T(k_1 + i - 1, \alpha)] \\ &\ge m\alpha[T(k_1 + 1, \alpha) - T(k_1, \alpha)] > m\, 2^{n-k_1-1}, \end{aligned}$$

which leads to a contradiction for $m \ge 2$. Thus, $T(n, \alpha)$ is attained at $k = k_1 + 1$.
Continuing the argument, we get the desired result.

Lemma 3.6 : For any $\alpha \ge 3$ fixed, $T(n, \alpha)$ is not attained at three (consecutive) values.

Proof : If possible, let $T(n, \alpha)$ be attained at the three values $k = K - 1, K, K + 1$. Then, using part (b) of Lemma 3.4, we get

$$\begin{aligned} 2^{n-K-1} &= \alpha[T(K+1, \alpha) - T(K, \alpha)] \\ &\ge \alpha[T(K, \alpha) - T(K-1, \alpha)] = 2^{n-K}, \end{aligned}$$

which is absurd.

From Lemma 3.5 and Lemma 3.6, we see that, for any $\alpha \ge 3$ fixed, $T(n, \alpha)$ is attained either at a unique k, or else at two (consecutive) values.

Lemma 3.7 : For some $\alpha \ge 3$, let $T(n, \alpha)$ be attained at $k = K, K + 1$. Then,

$$\alpha[T(K+1, \alpha) - T(K, \alpha)] = 2^{n-K-1}.$$

Proof : The proof follows immediately from the fact that

$$T(n, \alpha) = \alpha T(K, \alpha) + S(n-K, 3) = \alpha T(K+1, \alpha) + S(n-K-1, 3).$$

Corollary 3.5 : If α is not of the form 2^i, then $T(n, \alpha)$ is attained at a unique k.

Proof : If $T(n, \alpha)$ is attained at two values $k = K, K+1$, then the result in Lemma 3.7 is violated if α is not of the form 2^i.

Lemma 3.8 : For any $\alpha \geq 3$ fixed (with $\alpha \leq n$), let $T(n,\alpha)$ and $T(n+1,\alpha)$ both be attained at $k = K$. Then,

$$T(n+1, \alpha) - T(n, \alpha) = 2^{n-K}. \qquad (3)$$

Moreover, in such a case, if α is not of the form 2^i then

- **(a)** $T(n-1, \alpha)$ is not attained at $k = K$.
- **(b)** $T(n+2, \alpha)$ is not attained at $k = K$.

Proof : (3) follows from (2) (with $k_1 = k_2 = K$).

We prove parts (a) and (b) assuming that α is not of the form 2^i.

(a) Let $T(n-1, \alpha)$ be attained at $k = K$, so that

$$T(n-1, \alpha) = \alpha T(K, \alpha) + S(n-K-1, 3)$$
$$\leq \alpha T(K-1, \alpha) + S(n-K, 3).$$

Now, since

$$T(n+1, \alpha) = \alpha T(K, \alpha) + S(n+1-K, 3)$$
$$< \alpha T(K+1, \alpha) + S(n-K, 3),$$

we have the following chain of inequalities :

$$\alpha[T(K+1,\alpha) - T(K,\alpha)] > 2^{n-K}$$
$$\geq 2\alpha[T(K, \alpha) - T(K-1, \alpha)],$$

which contradicts part (b) of Lemma 3.4.

In this case, $T(n-1, \alpha)$ is attained at the (unique) point $k = K-1$, with

$$2^{n-K-1} < T(n,\alpha) - T(n-1,\alpha)$$
$$= \alpha[T(K,\alpha) - T(K-1,\alpha)] < 2^{n-K}.$$

(b) Let $T(n+2,\alpha)$ be attained at $k = K$. Then,

$$T(n+2,\alpha) = \alpha T(K, \alpha) + S(n+2-K, 3)$$
$$\leq \alpha T(K+1, \alpha) + S(n+1-K, 3).$$

Now, since

$$T(n,\alpha) = \alpha T(K, \alpha) + S(n-K, 3)$$
$$< \alpha T(K-1, \alpha) + S(n+1-K, 3),$$

we get

$$\alpha[T(K+1,\alpha) - T(K,\alpha)] \geq 2^{n+1-K}$$
$$> 2\alpha[T(K, \alpha) - T(K-1, \alpha)],$$

contradicting part (b) of Lemma 3.4.

Thus, $T(n+2, \alpha)$ is attained at the (unique) point $k = K+1$, satisfying the following chain of relations.

$$2^{n-K} < T(n+2,\alpha) - T(n+1,\alpha)$$
$$= \alpha[T(K+1,\alpha) - T(K,\alpha)] < 2^{n+1-K}.$$

All these complete the proof of the lemma.

Lemma 3.9 : Let, for some $\alpha \geq 3$, $T(n, \alpha)$ be attained at $k = K, K+1$. Then,

(a) $T(n-1, \alpha)$ is attained at $k = K$,

(b) $T(n+1, \alpha)$ is attained at $k = K+1$,

(c) $T(n, \alpha) - T(n-1, \alpha) = 2^{n-K-1} = T(n+1, \alpha) - T(n, \alpha)$.

Proof : Let $T(n, \alpha)$ be attained at $k = K, K+1$. Then,

$$T(n,\alpha) = \alpha T(K,\alpha) + S(n-K, 3)$$
$$= \alpha T(K+1,\alpha) + S(n-K-1, 3)$$
$$< \alpha T(K-1,\alpha) + S(n-K+1, 3).$$

(a) Let $T(n-1, \alpha)$ be attained not at $k = K$. Then, it must be attained at $k = K-1$. Thus,

$$T(n-1,\alpha) = \alpha T(K-1,\alpha) + S(n-K, 3)$$
$$< \alpha T(K,\alpha) + S(n-K-1, 3).$$

But, then

$$\alpha[T(K+1,\alpha) - T(K,\alpha)] = 2^{n-K-1}$$
$$< \alpha[T(K,\alpha) - T(K-1,\alpha)],$$

which contradicts part (b) of Lemma 3.4.

Thus, $T(n-1, \alpha)$ is attained at the (unique) point $k = K$.

(b) If, on the contrary, $T(n+1,\alpha)$ is attained at $k = K+2$, then

$$T(n+1,\alpha) = \alpha T(K+2,\alpha) + S(n-K-1, 3)$$
$$< \alpha T(K+1,\alpha) + S(n-K, 3).$$

Therefore,

$$\alpha[T(K+2,\alpha) - T(K+1,\alpha)] < 2^{n-K-1}$$
$$= \alpha[T(K+1,\alpha) - T(K,\alpha)],$$

and we are led to a contradiction to part (b) of Lemma 3.4.

Hence, $T(n+1, \alpha)$ is attained at the (unique) point $k = K+1$.

(c) follows from the proofs of parts (a) and (b).

For any $\alpha \geq 3$ and $n \geq 1$ fixed, let

$$T_\alpha(n,k) = \alpha T(k,\alpha) + S(n-k, 3) \text{ for } 0 \leq k \leq n-1.$$

Then,

$$T(n,\alpha) = \min_{0 \leq k \leq n-1} \left\{ T_\alpha(n,k) \right\}.$$

Lemma 3.10 : $T_\alpha(k)$ is convex in k in the sense that

$$T_\alpha(n,k+2) - T_\alpha(n,k+1) \geq T_\alpha(n,k+1) - T_\alpha(n,k) \text{ for all } 0 \leq k \leq n-2.$$

Proof : Since

$$T_\alpha(n, k+2) - T_\alpha(n, k+1)$$
$$= \alpha[T(k+2, \alpha) - T(k+1, \alpha)] - S(n-k-2, 3),$$

$$T_\alpha(n, k+1) - T_\alpha(n, k)$$
$$= \alpha[T(k+1, \alpha) - T(k, \alpha)] - S(n-k-1, 3),$$

we get

$$[T_\alpha(n, k+2) - T_\alpha(n, k+1)] - [T_\alpha(n, k+1) - T_\alpha(n, k)]$$
$$= \alpha[\{T_\alpha(n, k+2) - T_\alpha(n, k+1)\} - \{T_\alpha(n, k+1) - T_\alpha(n, k)\}]$$
$$+2^{n-k-2}.$$

The result now follows by virtue of part (b) Lemma 3.4.

4. DISCUSSION

When $\alpha = 2$, it can be proved that (see, for example, Majumdar [3])

$$T(n+1, 2) - T(n, 2) = 2^s \text{ for some integer } s \geq 1.$$

However, for $\alpha \geq 3$, such a relationship need not hold, as can be verified from the entries of Table 1, which gives the values of $T(n, \alpha)$ for $1 \leq n \leq 9$ and $3 \leq \alpha \leq 9$.

Table 1. Values of $T(n, \alpha)$ for $1 \leq n \leq 9$ and $3 \leq \alpha \leq 9$. In each cell, the number in parathesis gives the value(s) of k at which $T(n, \alpha)$ (in the formulation of (2.3)) is attained.

α \ n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	1 (0)	3 (0)	6 (1)	10 (1)	16 (2)	24 (2)	33 (3)	45 (4)	61 (4)
4	1 (0)	3 (0)	**7** (0, 1)	11 (1)	**19** (1, 2)	27 (2)	**43** (2, 3)	**59** (3, 4)	75 (4)
5	1 (0)	3 (0)	7 (0)	12 (1)	20 (1)	30 (2)	46 (2)	66 (3)	91 (4)
6	1 (0)	3 (0)	7 (0)	13 (1)	21 (1)	33 (2)	49 (2)	73 (3)	105 (3)
7	1 (0)	3 (0)	7 (0)	14 (1)	22 (1)	36 (2)	52 (2)	80 (3)	112 (3)
8	1 (0)	3 (0)	7 (0)	**15** (0, 1)	23 (1)	**39** (1, 2)	55 (2)	**87** (2, 3)	119 (3)
9	1 (0)	3 (0)	7 (0)	15 (0)	24 (1)	40 (1)	58 (2)	90 (2)	126 (3)

For $\alpha \geq 3$, it is an interesting problem to find the value(s) of n such that

$$T(n+1, \alpha) - T(n, \alpha) = 2^s \text{ for some integer } s \geq 1.$$

In this connection, we have the following result :

Lemma 4.1 : Let α be of the form 2^i (for some integer $i \geq 1$). Then, for all $n \geq 1$, the difference $T(n+1,\alpha) - T(n,\alpha)$ is of the form 2^s.

Proof : The proof is by induction on n. Corollary 2.1 shows that the result is true for $n = 2$. So, we assume the validity of the result for some n (so that the result is true for all i with $2 \leq i \leq n$). We have to show the validity of the result for $n + 1$.

To do so, let $T(n, \alpha)$ be attained at $k = K$. Then, one of the following two cases arises :

Case 1 : When $T(n + 1, \alpha)$ is attained at $k = K$.

In this case,

$$T(n+1,\alpha) - T(n, \alpha) = 2^{n-K}.$$

Case 2 : When $T(n + 1, \alpha)$ is attained at $k = K + 1$.

In this case,

$$T(n+1,\alpha) - T(n, \alpha) = \alpha[T(K + 1, \alpha) - T(K, \alpha)].$$

Then, by virtue of the induction hypothesis, $T(n+1, \alpha) - T(n, \alpha)$ is of the form 2^s for some integer $s \geq 1$.

If α be of the form 2^i, $T(n+1,\alpha) - T(n,\alpha)$ is of the form 2^s in the trivial case when $T(n + 1, \alpha) = S(n + 1, \alpha)$ (see Corollary 3.2). Thus,

$$T(2,\alpha) - T(1, \alpha) = 2 \text{ for all } \alpha \geq 3,$$

$$T(3,\alpha) - T(2, \alpha) = 2^2 \text{ for all } \alpha \geq 4.$$

Lemma 4.2 : Let α (≥ 3) be an integer, not of the form 2^i. Then $T(n+1,\alpha) - T(n,\alpha)$ is of the form 2^s (for some integer $s \geq 1$) if and only if $T(n+1, \alpha)$ and $T(n, \alpha)$ both are attained at the same $k = K$.

Proof : The "if" part of the lemma follows from Lemma 3.8. To prove the "only if" part, let $T(n, \alpha)$ be attained at $k = K$ and $T(n + 1, \alpha)$ be attained at $k = K+1$. Then,

$$T(n,\alpha) = \alpha T(K, \alpha) + S(n - K, 3)$$
$$< \alpha T(K + 1, \alpha) + S(n - K - 1, 3),$$
$$T(n+1,\alpha) = \alpha T(K + 1, \alpha) + S(n - K, 3)$$
$$< \alpha T(K, \alpha) + S(n + 1 - K, 3).$$

Then, we get the following chain of inequalities :

$$2^{n-K-1} < T(n+1, \alpha) - T(n, \alpha) < 2^{n-K}.$$

The above inequality shows that $T(n+1, \alpha) - T(n, \alpha)$ can not be of the form 2^s (for some integer $s \geq 1$).
Let α (≥ 3) be not of the form 2^i. Let for any α fixed, $T(n,\alpha)$ and $T(n + 1, \alpha)$ be such that $T(n+1,\alpha) - T(n, \alpha)$ is of the form 2^s (for some integer $s \geq 1$). From Lemma 3.8, coupled with Lemma 4.2, we see that, $T(n+2,\alpha) - T(n+1,\alpha)$ is not of the form 2^s. This raises the following question : Is there any integer N with $N > n$, such that $T(N+1,\alpha) - T(N,\alpha)$ is of the form 2^t (for some integer $t > s$)? The following lemma answers the question in the affirmative.

Lemma 4.3 : Let $\alpha = 2^i$ for some integer i (≥ 1). Let $T(n, \alpha)$ be attained at the two values $k = N, N + 1$. Let m be such that

$$m - n = n - N + i - 1. \tag{4}$$

Then, $T(m, \alpha)$ is attained at the two values $k = n-1, n$.

Proof : Let $T(n, \alpha)$ be attained at the two values $k = N, N+1$, so that by Lemma 3.9,

$$T(n, \alpha) - T(n-1, \alpha) = 2^{n-N-1} = T(n+1, \alpha) - T(n, \alpha).$$

Then,

$$\alpha[T(n, \alpha) - T(n-1, \alpha)] = 2^{m-n},$$

so that

$$\alpha T(n-1, \alpha) + S(m-n+1, 3) = \alpha T(n, \alpha) + S(m-n, 3).$$

Also, since

$$\alpha[T(n+1, \alpha) - T(n, \alpha)] = 2^{m-n} > 2^{m-n-1},$$

it follows that

$$\alpha T(n+1, \alpha) + S(m-n-1, 3) > \alpha T(n, \alpha) + S(m-n, 3).$$

Hence,

$$\begin{aligned} T(m, \alpha) &= \alpha T(n-1, \alpha) + S(m-n+1, 3) \\ &= \alpha T(n, \alpha) + S(m-n, 3) \\ &< \alpha T(n+1, \alpha) + S(m-n-1, 3), \end{aligned}$$

which shows that $T(m, \alpha)$ is attained at the two values $k = n-1, n$.

Lemma 4.3 shows that the sequence $\{T(n, \alpha)\}_{n=1}^{\infty}$ contains an infinite number of functions of the form $T(m, \alpha)$, each of which is attained at exactly two values of k.

When $\alpha = 2$, it can be shown (as in Majumdar [3]) that exactly one of the following two alternatives holds true :

Case (1) $T(n+2, \alpha) - T(n+1, \alpha) = T(n+1, \alpha) - T(n, \alpha)$, (5)

Case (2) $T(n+2, \alpha) - T(n+1, \alpha) = 2[T(n+1, \alpha) - T(n, \alpha)]$. (6)

When $\alpha(\geq 3)$ is of the form 2^i $(i \geq 2)$ $T(n+2,\alpha)$, $T(n+1,\alpha)$ and $T(n,\alpha)$ still satisfy one of the above two relationships. We now look at this problem more closely. Let $T(n, \alpha)$ be attained at two values $k = N$, $N+1$. By Corollary 3.5, α must be of the form 2^s (for some integer $s \geq 1$). From the proof of Lemma 3.9, we see that the necessary conditions (that $T(n, \alpha)$ is attained at $k = N$, $N+1$) are

1. $T(n, \alpha) - T(n-1, \alpha) = 2^{n-N-1}$,
2. $T(n+1, \alpha) - T(n, \alpha) = 2^{n-N-1}$,
3. $\alpha[T(N+1, \alpha) - T(N, \alpha) = 2^{n-N-1}$.

Now, we consider the function $T(n+2,\alpha)$. One of the following two cases arises :

Case 1 : When $T(n+2, \alpha)$ is attained at $k = N+1$.
In this case,

$$T(n+2, \alpha) - T(n+1, \alpha) = 2^{n-N},$$

so that

$$T(n+2, \alpha) - T(n+1, \alpha) = 2[T(n+1, \alpha) - T(n, \alpha)].$$

Case 2 : When $T(n+2, \alpha)$ is attained at $k = N+2$.
Here,

$$\begin{aligned} T(n+2,\alpha) &= \alpha T(N+2, \alpha) + S(n-N, 3) \\ &< \alpha T(N+1, \alpha) + S(n-N+1, 3), \end{aligned}$$

and hence,

$$\alpha[T(N+2,\alpha)-T(N+1,\alpha)]>2^{n-N}. \tag{7}$$

Then, $T(n+1,\alpha)$ is attained at $k=N+2$; otherwise,

$$T(n+1,\alpha)=\alpha T(N+1,\alpha)+S(n-N,3)$$
$$<\alpha T(N+2,\alpha)+S(n-N-1,3),$$

giving

$$\alpha[T(N+2,\alpha)-T(N+1,\alpha)]>2^{n-N-1},$$

which, together with Lemma 4.1, contradicts (7).

Hence, $T(n+1,\alpha)$ is attained at $k=N+2$, and consequently,

$$T(n+2,\alpha)-T(n+1,\alpha)=2^{n-N-1}.$$

When $\alpha(\geq 3)$ is not of the form 2^i $(i\geq 2)$, then $T(n+2,\alpha)$, $T(n+1,\alpha)$ and $T(n,\alpha)$ need not satisfy either of the two relationships (5) and (6). For example,

$$4=T(4,3)-T(3,3)<6=T(5,3)-T(4,3)<8=2[T(4,3)-T(3,3)].$$

We conjecture that, in such a case, neither of the relationships (4) and (5) holds true.

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