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Research Article

Integral Form of Popoviciu Inequality for Convex Function

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Abstract: In this paper, the new integral form of Popoviciu inequality for convex functions is constructed and also the new refinement of integral form of Jensen's inequality is given. For the purpose of application some new quasi arithmetic means are defined along with their monotonicity property.

Keywords: Convex function, Popoviciu inequality, Jensen's inequality, quasi arithmetic means

1. INTRODUCTION AND PRELIMINARY RESULTS

A function $g: C \to \mathbb{R}$, where C is a convex subset of real vector space, is said to be convex if

$$g(ax+by) \leq ag(x)+bg(y), \tag{1}$$

for all $x, y \in C$ and $a, b \geq 0$, such that $a+b=1$ (see [10, page ~1]).

In [10, page ~43] the Jensen's inequality in discrete version is given as follows:

Theorem 1.1 *Let C be a convex subset of real vector spaceX, $g: C \to \mathbb{R}$ be convex function, $p_1, \ldots, p_n \in (0,1]$ such that $\sum_{i=1}^{n} p_i = 1$, and $c_1, \ldots, c_n \in C$, then*

$$g\left(\sum_{i=1}^{n} p_i c_i\right) \leq \sum_{i=1}^{n} p_i g(c_i). \tag{2}$$

In [10, page ~63] the integral form of Jensen's inequality is defined as follows.

Theorem 1.2 *Let h be an integrable function on a probability space (X, A, μ) taking values in an interval $I \subset \mathbb{R}$. If g is a convex function on I such that the composition function $g \circ h$ is integrable, then*

$$g\left(\int_X h d\mu\right) \leq \int_X g \circ h d\mu. \tag{3}$$

In [2], Brnetić, Pearce and Pečaric give the refinement of integral form of Jensen's inequality (3) by using refinement of discrete Jensen's inequality. Moreover in [7], László Horváth and Pečarić give

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the improvement of integral form of Jensen's inequality (3) by using some refinement of discrete Jensen's inequality which is generalization of result given in [2], they also give new quasi arithmetic means and prove their monotonicity.

The Popoviciu inequality is given by (see [10, page 173]).

Theorem 1.3 *Let* $m,n\in\mathbb{N}$*, such that* $n\geq 3$*,* $2\leq m\leq n-1$*,* $[a,b]\subset\mathbb{R}$ *be an interval,* $\mathbf{x}=(x_1,\ldots,x_n)\in[a,b]^n$*,* $\mathbf{p}=(p_1,\ldots,p_n)$ *be a* n*-tuple such that* $p_i\geq 0, i=1,2,\ldots,n$ *with* $\sum_{i=1}^{n}p_i=1$*. Also let* $g:[a,b]\to\mathbb{R}$ *be a convex function. Then*

$$g_{m,n}(\mathbf{x},\mathbf{p})\leq\frac{n-m}{n-1}g_{1,n}(\mathbf{x},\mathbf{p})+\frac{m-1}{n-1}g_{n,n}(\mathbf{x},\mathbf{p}), \tag{4}$$

where

$$g_{m,n}:=\frac{1}{C_{m-1}^{n-1}}\sum_{1\leq i_1<\ldots<i_m\leq n}\left(\sum_{j=1}^{m}p_{i_j}\right)g\left(\frac{\sum_{j=1}^{m}p_{i_j}x_{i_j}}{\sum_{j=1}^{m}p_{i_j}}\right), \tag{5}$$

and

$$C_m^n=\frac{n!}{m!(n-m)!}.$$

In the current century, the Popoviciu inequality (4) is studied by many authors. In the monograph [6], the generalization of (4) for real weights, mixed symmetric means, exponential convexity, mean value theorems and Cauchy means are studied. In [8, 9], the integral version and refinement of a special case of (4) is proved respectively. In [1], the higher dimension analogue of a special case of (4) is given. Moreover, in [3, 4, 5] (4) is generalized for higher order convex functions via different interpolating polynomials. We use the idea of Brnetić, Pearce and Pečarić given (for Jensen's inequality) in [2] to construct the integral form of Popoviciu inequality (4). Also following the way of László Horváth and J. Pečarić given (for refinements of Jensen's inequality) in [7] we give application to the quasi arithmetic means.

2. MAIN RESULTS

We now consider some hypotheses which are used in our work.

(H_1) Let $(X,\mathcal{E},\mu)$ be a probability space, and let $p_1,\ldots,p_n$ be positive numbers with $\sum_{i=1}^{n}p_i=1$.

(H_2) Let $h:X\to I\subset\mathbb{R}$ be an integrable function.

(H_3) Let g be a convex function on interval I such that the composition $g\circ h$ is integrable.

Let $m \geq 2$ be a fixed integer. The σ-algebra in X^k generated by the projection mapping $pr_l : X^k \to X \; (l = 1, \ldots, m)$

$$pr_l(x_1, \ldots, x_m) := x_l \tag{6}$$

is denoted by E^k. And μ^m is defined as the product measure on E, this measure is uniquely (μ is σ-finite) specified by

$$\mu^m(B_1 \times \ldots \times B_m) := \mu(B_1) \ldots \mu(B_m), \quad B_l \in \mathsf{E}, \quad l = 1, \ldots, m. \tag{7}$$

Theorem 2.1 *Assume* (H_1) - (H_3)*, then the following inequalities hold.*

a.

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right) \int_{X^m} g \left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} \right) d\mu^m(x_{i_1}, \ldots, x_{i_m})$$

$$\leq \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_X g(h(x_i)) d\mu(x_i)$$

$$+ \frac{m-1}{n-1} \int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i) \right) d\mu^n(x_{i_1}, \ldots, x_{i_n}).$$

b.

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right) \int_{X^m} g \left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} \right) d\mu^m(x_{i_1}, \ldots, x_{i_m})$$

$$\leq \sum_{i=1}^{n} p_i \int_X g(h(x_i)) d\mu(x_i).$$

Proof. (a) On integrating the inequality (4) over X^n and replacing x_{i_j} by $h(x_{i_j})$, we have

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right) \int_{X^n} g \left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} \right) d\mu^n(x_{i_1}, \ldots, x_{i_n})$$

$$\leq \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_{X^n} g(h(x_i)) d\mu^n(x_{i_1}, \ldots, x_{i_n})$$

$$+\frac{m-1}{n-1}\int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

On simplification we have

$$\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m} p_{i_j}\right)\int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$

$$\times\int_X d\mu(x_{i_{m+1}})\ldots\int_X d\mu(x_{i_n})$$

$$\le\frac{n-m}{n-1}\sum_{i=1}^{n} p_i\int_X g\circ h(x_i)d\mu(x_i)\times\int_X d\mu(x_{i_1})\ldots\int_X d\mu(x_{i_m})\int_X d\mu(x_{i_{m+1}})\ldots\int_X d\mu(x_{i_n})$$

$$+\frac{m-1}{n-1}\int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

This gives

$$\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m} p_{i_j}\right)\int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$

$$\le\frac{n-m}{n-1}\sum_{i=1}^{n} p_i\int_X g(h(x_i))d\mu(x_i)$$

$$+\frac{m-1}{n-1}\int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

(b) Using the discrete Jensen's inequality in the last term of inequality given in (a) and on solving, we have

$$\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m} p_{i_j}\right)\int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$

$$\le\frac{n-m}{n-1}\sum_{i=1}^{n} p_i\int_X g(h(x_i))d\mu(x_i)$$

$$+\frac{m-1}{n-1}\left(p_1\int_{X^n} g(h(x_1))d\mu^n(x_{i_1},\ldots,x_{i_n})+\ldots+p_n\int_{X^n} g(h(x_n))d\mu^n(x_{i_1},\ldots,x_{i_n})\right),$$

this gives

$$\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)\int_{X^m} g\left(\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}\right)d\mu^m(x_{i_1},\ldots,x_{i_m})$$

$$\le \sum_{i=1}^{n}p_i\int_X g(h(x_i))d\mu(x_i).$$

Under the hypothesis (H_1), (H_2) and (H_3), define the function $H_m(t)$ on $[0,1]$ given by

$$H_m(t)=\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times\int_{X^m} g\left(t\,\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-t)\int_X h d\mu\right)d\mu^m(x_{i_1},\ldots,x_{i_m}). \tag{8}$$

Theorem 2.2 *For* $m\ge 2$ *be an integer, we assume* (H_1) - (H_3) *and consider* H_m $:[0,1]\to R$ as defined in (8) then the following statements are valid.

a. H_m is convex.

b. $\min_{t\in[0,1]} H_m(t)=H_m(0)=g\left(\int_X h d\mu\right)$

c. $\max_{t\in[0,1]} H_m(t)=H_m(1)$

d. H_m is increasing.

Proof. (a) Suppose $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and $u,v\in[0,1]$, then from (8) we have

$$H_m(\alpha u+\beta v)=\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times\int_{X^m} g\left((\alpha u+\beta v)\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(\alpha+\beta-\alpha u-\beta v)\int_X h d\mu\right)d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

On simplification we have

$$H_m(\alpha u+\beta v)=\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times \int_{X^m} g(\alpha\left(u\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-u)\int_X h d\mu \right) +$$

$$\beta\left(v\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-v)\int_X h d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

By convexity of g, we have

$$H_m(\alpha u+\beta v) \leq \alpha \frac{1}{C_{m-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_m\leq n} \left(\sum_{j=1}^{m} p_{i_j}\right)$$

$$\times \int_{X^m} g\left(u\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-u)\int_X g d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m}) +$$

$$\beta \frac{1}{C_{m-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_m\leq n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(v\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-v)\int_X h d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m}),$$

that is

$$H_m(\alpha u+\beta v) \leq \alpha H_m(u) + \beta H_m(v).$$

Therefore H_m is convex function.

(b) By the integral from Jensen's inequality (8) yields

$$H_m(t) \geq \frac{1}{C_{m-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_m\leq n} \left(\sum_{j=1}^{m} p_{i_j}\right)$$

$$\times g\left(\int_{X^m} \left(t\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-t)\int_X g d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m}) \right)$$

or

$$H_m(t) \geq \frac{1}{C_{m-1}^{n-1}} \sum_{1\leq i_1<\ldots<i_m\leq n} \left(\sum_{j=1}^{m} p_{i_j}\right) g(I), \tag{9}$$

where

$$I = \int_{X^m} \left(t \frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-t)\int_X g d\mu \right) d\mu^m (x_{i_1},\ldots,x_{i_m})$$

$$= t \int_{X^m} \frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} d\mu^m (x_{i_1},\ldots,x_{i_m}) + (1-t) \int_{X^m} \left(\int_X g d\mu \right) d\mu^m (x_{i_1},\ldots,x_{i_m})$$

$$= \frac{t}{\sum_{j=1}^{m} p_{i_j}} \sum_{i=1}^{m} p_i \int_X h d\mu + (1-t)\int_X g d\mu$$

$$= \int_X g d\mu$$

so from (9), we have

$$H_m(t) \geq \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right) g\left(\int_X g d\mu \right)$$

$$= H_m(0), \qquad \forall\, t \in [0,1].$$

(c)

$$H_m(t) = H_m(1.t + (1-t)0) \leq tH_m(1) + (1-t)H_m(0)$$

$$\leq tH_m(1) + (1-t)H_m(1)$$

$$= H_m(1), \quad \forall\, t \in [0,1].$$

(d) Since $H_m(t)$ is convex and $H_m(t) \geq H_m(0)(t \in [0,1])$, therefore for $0 \leq t_1 < t_2 \leq 1$, we have

$$\frac{H_m(t_2) - H_m(t_1)}{t_2 - t_1} \geq \frac{H_m(t_2) - H_m(0)}{t_2} \geq 0,$$

so

$$H_m(t_2) \geq H_m(t_1).$$

Theorem 2.3 *Assume* (H_1), (H_2) *and* (H_3), *then*

$$g\left(\int_X h d\mu \right) \leq H_m(t) \leq H_m(1) \leq \int_X g \circ h d\mu. \tag{10}$$

Proof. Using (b) and (c) of Theorem 2.2 we get first two inequalities, and for the last inequality

$$H_m(1) = \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right)$$

$$\times \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

Using the discrete Jensen's inequality, we have

$$H_m(1) \le \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right)$$

$$\times \frac{\sum_{j=1}^{m} p_{i_j}}{\sum_{j=1}^{m} p_{i_j}} \int_{X^m} g\left(h(x_{i_j})\right) d\mu^m(x_{i_1},\ldots,x_{i_m}),$$

this gives

$$H_m(1) \le \int_X g \circ h d\mu.$$

Remark 2.4 *A refinement similar to (10) of integral form of Jensen's inequality is proved in Proposition 7 of [7].*

3. NEW QUASI-ARITHMETIC MEANS

Now we introduced some new quasi arithmetic means. For this first assume some conditions:

(H_4) Let $h: X \to I$, where $I \subset \mathbb{R}$ be an interval, is measurable.

(H_5) Let α, $\beta: I \to \mathbb{R}$ are continuous and strictly monotone functions.

Definition 1 *Assume* (H_1), (H_4) *and* (H_5).

For $t \in [0,1]$ we define the class of quasi-arithmetic mean given by

$$M_{\lambda,\chi}(t,g,\mu) := \chi^{-1}\Big(\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right)$$

$$\times \int_{X^m} \chi \circ \lambda^{-1}\left(t \frac{\sum_{j=1}^{m} p_{i_j} \lambda(g(x_{i_j}))}{\sum_{j=1}^{m} p_{i_j}} + (1-t)\int_X \lambda(g) d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m})\Big), \quad (11)$$

where the integrals are supposed to be exist.

Assume (H_6), let $\eta: I \to \mathbb{R}$ be a continuous and strictly monotone function such that the composition $\eta \circ h$ is integrable on X. Define the mean

$$M_\eta(h,\mu) = \eta^{-1}\left(\int_X \eta \circ h d\mu\right). \quad (12)$$

Theorem 3.1 *Assume* (H_1), (H_4), (H_5) *and assume that* $\lambda \circ h$ *and* $\chi \circ h$ *an integrable on* X.

(a) If $\chi \circ \lambda^{-1}$ is convex with χ is increasing or $\chi \circ \lambda^{-1}$ is concave with χ is decreasing, then

$$M_\lambda(h,\mu) \le M_{\chi,\lambda}(t,h,\mu) \le M_\chi(h,\mu), \tag{13}$$

holds for all $t \in [0,1]$.

(b) If $\chi \circ \lambda^{-1}$ is convex with χ is decreasing or $\chi \circ \lambda^{-1}$ is concave with χ is increasing, then

$$M_\lambda(h,\mu) \ge M_{\chi,\lambda}(t,h,\mu) \ge M_\chi(h,\mu), \tag{14}$$

holds for all $t \in [0,1]$.

Proof. (a) Using pair of functions $\chi \circ \lambda^{-1}$ and $\lambda(h)$ ($\lambda(I)$ is an interval) in Theorem 2.3, we have

$$\chi \circ \lambda^{-1}\left(\int_X \lambda(h)d\mu\right) \le \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^m p_{i_j}\right)$$
$$\times \int_{X^m} \chi \circ \lambda^{-1}\left(t\frac{\sum_{j=1}^m p_{i_j}\lambda(h(x_{i_j}))}{\sum_{j=1}^m p_{i_j}} + (1-t)\int_X \lambda(h)d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^m p_{i_j}\right) \int_{X^m} \chi \circ \lambda^{-1}\left(\frac{\sum_{j=1}^m p_{i_j}\lambda(h)}{\sum_{j=1}^m p_{i_j}} \right) d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

Using the discrete Jensen inequality on the right side of last inequality we get

$$\chi \circ \lambda^{-1}\left(\int_X \lambda(h)d\mu\right) \le \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^m p_{i_j}\right)$$
$$\times \int_{X^m} \chi \circ \lambda^{-1}\left(t\frac{\sum_{j=1}^m p_{i_j}\lambda(h(x_{i_j}))}{\sum_{j=1}^m p_{i_j}} + (1-t)\int_X \lambda(h)d\mu \right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \int_X \chi(h)d\mu.$$

On taking χ^{-1} on both sides we have (13).

(b) Similarly using the pair of functions $-\chi \circ \lambda^{-1}$ and $\lambda(h)$ in Theorem 2.3, where $\chi \circ \lambda^{-1}$ is concave. On taking χ^{-1} the we have (14).

4. REFERENCES

1. Bencze, M., C.P. Niculescu & F. Popovici. Popovicius inequality for functions of several variables. *Journal of Mathematical Analysis and Applications* 365: 399-409 (2010).
2. Brnetić, I., C.E. M. Pearce & J. Pečarić. Refinements of Jensen's inequality. *Tamkang Journal of Matematics* 31: 63-69 (2000).
3. Butt, S.I., K.A.Khan, & J. Pečarić. Generalization Of Popoviciu Inequality For Higher Order Convex Functions Via Taylor Polynomial. *Acta Universitatis Apulensis* 42: 181-200 (2015).
4. Butt,S. I.,K. A.Khan & J.Pečaric. Popoviciu type inequalities via Green function and generalized Montgomery identity. *Mathematical Inequalities and Applications* 18: 1519-1538 (2015).
5. Butt,S. I., K. A. Khan & J. Pečarić. Popoviciu type inequalities via Green function and Taylor polynomial. *Turkish Journal of Mathematics* 40: 333–349 (2016).
6. Horváth,L., K.A. Khan & J. Pečarić. Combinatorial Improvements of Jensens Inequality / Classical and New Refinements of Jensens Inequality with Applications. *Monographs in Inequalities 8*, Zagreb: Element (2014).
7. Horváth, L. & J. Pečarić. Refinement of the classical Jensen's inequality coming from refinement of the discrete Jensen's inequality. *Advances in Inequalities and Applications* 3(3): 323-328 (2015).
8. Niculescu, C. P. The integral version of Popoviciu's inequality. *Journal of Mathematical Inequalities* 3(3): 323-328 (2009).
9. Niculescu, C.P. & F. Popovici. A refinement of Popovicius inequality. *Bulletin of the Society for Mathematical Sciences in Romania* 49: 285-290 (2006).
10. Pečarić, J., F. Proschan & Y.L. Tong. *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York (1992).