Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 53 (3): 325–337 (2016)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

On Further Study of CA-AG-groupoids

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Abstract: An AG-groupoid that satisfies the identity a(bc) = c(ab) is called a CA-AG-groupoid [1]. In this article various properties of CA-AG-groupoids are explored and their relations with various other known subclasses of AG-groupoids and with some other algebraic structures are established. We proved that in CA-AG-groupoid left alternativity implies right alternativity and vice versa. We also proved that a CA-AG-groupoid having a right cancellative element is a T^1, a T^3 and an alternative AG-groupoid. We provided a partial solution to an open problem of right cancellative element of an AG-groupoid. Further, we proved that a CA-AG-groupoid having left identity is a commutative semigroup and investigated that the direct product of any two CA-AG-groupoids is again cyclic associative. Moreover, we investigated relation among CA, AG* and Stein AG-groupoids.

Keywords: AG-groupoid, CA-AG-groupoid, bi-commutative, Stein AG-groupoids, direct product

1. INTRODUCTION

A groupoid satisfying the "left invertive law" is called an Abel-Grassmann's groupoid (or simply AG-groupoid [2]). In literature different names like "left almost semigroup" (LA-semigroup) [3], left invertive groupoid [4] and right modular groupoid [5] has been used by different authors for the said structure. Many properties of AG-groupoids have been studied in [6, 7]. Various aspects of AG-groupoids have been studied in [2, 8 − 14]. In [1,15], we introduced CA-AG-groupoid as a new subclass of AG-groupoid and studies some fundamental properties of it. In the same paper we introduced CA-test for the verification of cyclic associativity for an arbitrary AG-groupoid. We also enumerated CA-AG-groupoids up to order 6 and further classified it into different subclasses.

2. PRELIMINARIES

A groupoid $(G,\cdot)$ or simply G satisfying the "left invertive law [3]: $(ab)c = (cb)a \ \forall a, b, c \in G$" is called an Abel-Grassmann's groupoid (or simply AG-groupoid [2]). Through out the article we will denote an AG-groupoid simply by S otherwise stated else. S always satisfies the "medial law: $(ab)(cd) = (ac)(bd)$ [16, Lemma 1.1(i)], while S with left identity e is called an AG-monoid and it always satisfies the paramedial law: $(ab)(cd) = (db)(ca)$ [16, Lemma 1.2(ii)]". A groupoid G is called right AG-groupoid or right almost semigroup (RA-semigroup) [3] if $\forall a, b, c \in G$, $a(bc) = c(ba)$. An AG-groupoid S is called:

i. cyclic associative AG-groupoid (CA-AG-groupoid) [15]; if $a(bc) = c(ab) \ \forall a, b, c \in S$.

ii. AG* [8]; if $(ab)c = b(ac)$.

iii. AG** [9]; if $a(bc) = b(ac)$.

iv. T^1-AG-groupoid [10] if $\forall a, b, c, d \in S$, $ab = cd$ implies $ba = dc$.

Received, November 2015; Accepted, July 2016
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vi. left T^3-AG-groupoid (T_l^3-AG-groupoid) if $\forall a, b, c \in S$, $ab = ac$ implies $ba = ca$.

vii. right T^3-AG-groupoid (T_r^3-AG-groupoid) if $ba = ca$ implies $ab = ac$.

viii. T^3-AG-groupoid [10] if it is T_l^3 as well as T_r^3.

ix. transitively commutative if $ab = ba$ and $bc = cb$ implies $ac = ca$ $\forall a, b, c \in S$.

x. Bol*-AG-groupoid [17] if it satisfies the identity $a(bc \cdot d) = (ab \cdot c)d$ $\forall a, b, c, d \in S$.

xi. left alternative if $\forall a, b \in S$, $(aa)b = a(ab)$ and is called right alternative if $b(aa) = (ba)a$. S is called alternative [10], if it is both left alternative and right alternative.

xii. An element $a \in S$ is left cancellative (resp. right cancellative) [17] if $\forall w, y \in S$, $aw = ax \Rightarrow w = x$ ($wa = xa \Rightarrow w = x$).

xiii. An element $a \in S$ is cancellative if it is both left and right cancellative. S is left cancellative (resp. right cancellative, cancellative) if every element of S is left cancellative (right cancellative, cancellative).

xiv. left commutative (resp. right commutative) if $\forall a, b, c \in S$, $(ab)c = (ba)c$ ($a(bc) = a(cb)$). S is called bi-commutative AG-groupoid [18], if it is left and right commutative.

xv. Stein-AG-groupoid [18], if $a(bc) = (bc)a$ $\forall a, b, c \in S$.

xvi. An element $a \in S$ is called idempotent if $a^2 = a$ and an AG-groupoid having each element as idempotent, is called AG-2-band (or simply AG-band) [12].

xvii. A groupoid in which $(ab)c = a(bc)$, $\forall a, b, c \in S$ holds is called a semigroup. If a semigroup contains the identity element e such that $ea = a = ae$, then it is called monoid.

Due to non-associativity of AG-groupoid, left identity does not imply right identity and so the identity.

For two AG-groupoids S_1 and S_2, the set $\{(a, b) | a \in S_1, b \in S_2\}$ with the "binary operation defined by $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ is called the direct product of S_1 and S_2, denoted by $S_1 \times S_2$", in this case we say that S_1 and S_2 are the direct factors of $S_1 \times S_2$.

3. VARIOUS PROPERTIES OF CA-AG-GROUPOIDS

In the following, it is observed that the subclass of CA-AG-groupoid is distinct from that of T^1 and T^3-AG-groupoids. We provide a counter example to show that a CA-AG-groupoid is not a T^1-AG-groupoid, however, a CA-AG-groupoid with a right cancellative element is (i) T^1-AG-groupoid and (ii) T^3-AG-groupoid.

Example 1. Table 1 represents a CA-AG-groupoid of order 4. As $4 \cdot 3 = 2 = 3 \cdot 3$ but $3 \cdot 4 \neq 3 \cdot 3$, thus it is not a T^1-AG-groupoid.

Table 1. CA-AG-groupoid that is not T^1.

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2

However, we have the following;

Theorem 1. *Every CA-AG-groupoid having a right cancellative element is a* T^1*-AG-groupoid.*

Proof. Let S be a CA-AG-groupoid having a right cancellative element z and $a, b, c, d \in S$. Let $ab = cd$, then by cyclic associativity, left invertive law and right cancellativity, we have;

$$z^2(ba) = a(z^2b) = a(zz \cdot b) = a(bz \cdot z) = z(a \cdot bz)$$
$$= z(z \cdot ab) = z(z \cdot cd) = z(d \cdot zc) = z(c \cdot dz)$$
$$= (dz)(zc) = c(dz \cdot z) = c(z^2d) = d(cz^2) = z^2(dc)$$
$$\Rightarrow z^2(ba) = z^2(dc) \Rightarrow (ba \cdot z)z = (dc \cdot z)z$$
$$\Rightarrow ba \cdot z = dc \cdot z \Rightarrow ba = dc.$$

Hence, S is a T^1-AG-groupoid.

Since a T^1-AG-groupoid is (i) a T^3-AG-groupoid [10], and (ii) an AG**-groupoid [11]. Thus, we immediately have the following corollary.

Corollary 1. Every cancellative CA-AG-groupoid or simply having a right cancellative element is

(i) a T^3-AG-groupoid.

(ii) an AG**-groupoid.

Lemma 1. *Every left cancellative CA-AG-groupoid S is transitively commutative.*

Proof. Let S be a left cancellative CA-AG-groupoid and $a, b, c \in S$ such that $ab = ba$ and $bc = cb$. We have to show that $ac = ca$. Using cyclic associativity and the assumption, we have $b(ac) = c(ba) = c(ab) = b(ca) \Rightarrow b(ac) = b(ca)$, which by left cancellativity imply $ac = ca$. Hence S is transitively commutative.

Now, we discuss an open problem given in [17] and provide a partial solution to that open problem. To this end, we first restate the following [17, Theorem 26].

Theorem 2. *"Every right cancellative element of an AG-groupoid S is (left) cancellative."*

The converse of the above theorem is not true in general. In 2012, M. Shah proposed an open Problem in his Ph.D thesis [17]: "Prove or disprove that in an AG-groupoid, without left identity, every left cancellative element is right cancellative". In [17], the open problem have been partially resolved by the proposer himself, that is: (a) "An AG-groupoid, a left cancellative element is right cancellative, if either S is cancellative or if S has left identity [17, Theorem 28]", (b) "In an AG-groupoid, a left cancellative element x is right cancellative if any of the following holds: (i) If x is idempotent (ii) If x^2 is left cancellative (iii) If there exists a left nuclear left cancellative element in S". The converse of the problem has also been proved for AG*-groupoid, AG**-groupoid and self-dual AG-groupoid i.e. (i) "every left cancellative element of an AG*-groupoid is right cancellative [17] (ii) every left cancellative element of an AG**-groupoid is right cancellative [17] (iii) every left cancellative element of self-dual AG-groupoid is right cancellative [19]". We claim that the converse of Theorem 2 also holds for CA-AG-groupoids and verify the claim in the following theorem.

Theorem 3. *Every left cancellative element of a CA-AG-groupoid is right cancellative.*

Proof. Let a be a left cancellative element of a CA-AG-groupoid S. To show that a is right cancellative, let $xa = ya$ for all $x, y \in S$. Then, by cyclic associativity, medial law and assumption, we have

$$a(a \cdot ax) = (ax)(aa) = (aa)(xa) = (aa)(ya)$$
$$= (ay)(aa) = a(ay \cdot a) = a(a \cdot ay)$$
$$\Rightarrow a(a \cdot ax) = a(a \cdot ay).$$

This by repeated use of the left cancellativity of a implies that $x = y$. Hence a is right cancellative.

Next we prove that any cancellative element of a CA-AG-groupoid can be written as the product of its two cancellative elements.

Theorem 4. *Every cancellative element of a CA-AG-groupoid can be written as the product of its two cancellative elements.*

Proof. Let a be an arbitrary cancellative element of a CA-AG-groupoid S. Suppose $a = c_1c_2$, where c_1 and c_2 are any arbitrary elements of S. We have to show that c_1 and c_2 are cancellative. Consider $xc_1 = yc_1$, then by cyclic associativity we have

$$xa = x(c_1c_2) = c_2(xc_1) = c_2(yc_1) = c_1(c_2y) = y(c_1c_2) = ya \Rightarrow xa = ya.$$

Which by the right cancellativity of a implies $x = y$. Thus c_1 is right cancellative and hence cancellative by Theorem 2. Now let $c_2x = c_2y$. Then

$$xa = x(c_1c_2) = c_2(xc_1) = c_1(c_2x) = c_1(c_2y) = y(c_1c_2) = ya$$

this by the right cancellativity of a implies that $x = y$. Thus c_2 is left cancellative and thus cancellative by Theorem 3. Hence the result follows.

Example 2. Table 2 represent a CA-AG-groupoid having 1 and 3 as cancellative elements, while 2 as non-cancellative element. 1 and 3 are the product to two cancellative elements.

Table 2. CA-AG-groupoid with two cancellative elements.

·	1	2	3
1	1	2	3
2	2	2	2
3	3	2	1

Theorem 5. *Let k be a fixed element of a CA-AG-groupoid S such that $ak = ka$ and $bk = kb$ for some a, b in S. If k is left or right cancellative then a, b commute.*

Proof. First assume that k is left cancellative. Then, using cyclic associativity and given condition,

$$k(ab) = b(ka) = a(bk) = a(kb) = b(ak) = k(ba)$$

which by left cancellativity of k implies that $ab = ba$.

Now, let k is right cancellative, then by Theorem 2, k is left cancellative and hence the result follows.

Theorem 6. *Every CA-AG-groupoid is paramedial* [1].

Next attention is paid towards alternative AG-groupoids. The following example shows that left alternative and right alternative are distinct subclasses of AG-groupoids.

Example 3. Left alternative AG-groupoid of order 4 given in Table 3 is not right alternative because, $a(bb) \neq (ab)b$.

Table 3. Left alternative AG-groupoid that is not right alternative.

·	a	b	c	d
a	c	c	d	d
b	d	b	d	d
c	d	d	d	d
d	d	d	d	d

The right alternative AG-groupoid of order 3 represented in Table 4 is not a left alternative AG-groupoid since $(1 \cdot 1)2 \neq 1(1 \cdot 2)$.

Table 4. Right alternative AG-groupoid that is not left alternative.

·	1	2	3
1	3	2	3
2	1	3	3
3	3	3	3

However, if an AG-groupoid is cyclic associative then left alternativity implies right alternativity and vice versa, as proved in the next theorem.

Theorem 7. *Let S be a CA-AG-groupoid, then S is left alternative if and only if S is right alternative.*

Proof. Assume first that S is a left alternative CA-AG-groupoid, then for any a, b in S

$$b \cdot aa = a \cdot ba = a \cdot ab = aa \cdot b = ba \cdot a$$
$$\Rightarrow b \cdot aa = ba \cdot a.$$

Conversely, assume that S is right alternative, then

$$aa \cdot b = ba \cdot a = b \cdot aa = a \cdot ba = a \cdot ab$$
$$\Rightarrow aa \cdot b = a \cdot ab.$$

Hence the theorem is proved.

In the following example it is shown that the class of CA-AG-groupoid is distinct from the class of alternative AG-groupoids.

Example 4. CA-AG-groupoid of order 4, presented in Table 5, is neither a left alternative nor a right alternative because $(4 \cdot 4)4 \neq 4(4 \cdot 4)$.

Table 5. CA-AG-groupoid that is not alternative AG-groupoid.

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	3

However, if a CA-AG-groupoid contains element either as a left or as a right cancellative then, it becomes an alternative AG-groupoid, as established in the following result.

Theorem 8. *A CA-AG-groupoid with a left cancellative element is an alternative AG-groupoid.*

Proof. Let S be a CA-AG-groupoid having a left cancellative (and hence a cancellative) element x and $a, b \in S$. Then by cyclic associativity and left invertive law:

$$x(aa \cdot b) = b(x \cdot aa) = b(a \cdot xa) = (xa)(ba)$$
$$= a(xa \cdot b) = a(ba \cdot x) = x(a \cdot ba) = x(a \cdot ab),$$

which by left cancellativity of x implies $(aa)b = a(ab)$. Thus S is left alternative AG-groupoid. By virtue of Theorem 7, S is also right alternative. Hence S is alternative.

The following example suggests that neither every cancellative AG-groupoid nor every alternative AG-groupoid is CA.

Example 5. Table 6, represents a cancellative AG-groupoid of order 3. As $3(2 \cdot 1) \neq 1(3 \cdot 2)$, hence it is not cyclic associative.

Table 6. Cancellative AG-groupoid that is not CA-AG-groupoid.

·	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table 7 represents an alternative AG-groupoid of order 4, which is not cyclic associative since $a(ba) \neq a(ab)$.

Table 7. Alternative AG-groupoid that is not CA-AG-groupoid.

·	a	b	c	d
a	c	c	c	b
b	d	c	c	c
c	c	c	c	c
d	c	a	c	c

Now, we demonstrate that the class of CA-AG-groupoid is distinct from the class of Stein AG-groupoid. To begin with, consider the following:

Example 6. CA-AG-groupoid of order 4, represented in Table 8, is not a Stein AG-groupoid as: $(1 \cdot 1)1 \neq 1(1 \cdot 1)$. While a Stein AG-groupoid of order 4, presented in Table 9, is not a CA-AG-groupoid since $1(1 \cdot 2) \neq 2(1 \cdot 1)$.

Table 8. CA-AG-groupoid that is not Stein AG-groupoid.

·	1	2	3	4
1	2	3	3	3
2	4	3	3	3
3	3	3	3	3
4	3	3	3	3

Table 9. Stein AG-groupoid that is not CA-AG-groupoid.

·	1	2	3	4	5
1	3	3	4	5	5
2	4	4	5	5	5
3	4	5	5	5	5
4	5	5	5	5	5
5	5	5	5	5	5

Further, the following example establish that neither every AG*-groupoid is Stein, nor every Stein AG-groupoid is AG*.

Example 7. Table 10, represents an AG*-groupoid of order 6. As $1(1 \cdot 2) \neq (1 \cdot 2)1$, hence it is not a Stein AG-groupoid. A Stein AG-groupoid of order 5 given in Table 9 of Example 6 is not an AG*-groupoid as $(1 \cdot 1)2 \neq 1(1 \cdot 2)$.

Table 10. AG*-groupoid that is not Stein AG-groupoid.

·	1	2	3	4	5	6
1	3	4	5	5	5	5
2	3	4	6	6	5	5
3	5	5	5	5	5	5
4	6	6	5	5	5	5
5	5	5	5	5	5	5
6	5	5	5	5	5	5

However, by coupling any two from CA, Stein and AG*-groupoids, we get the third one. As proved in the following;

Theorem 9. *Let S be an AG-groupoid then, any two of the following implies the third one.*

(i) *S is CA.*

(ii) *S is AG*.*

(iii) *S is Stein.*

Proof. Let S be an AG-groupoid and $a, b, c \in S$.

(i) and (ii) implies (iii): Using the properties of cyclic associativity, definition of AG* and the left invertive law, $a(bc) = c(ab) = b(ca) = (cb)a = (ab)c = b(ac) = c(ba) = (bc)a$. Hence S is a Stein AG-groupoid.

(ii) and (iii) implies (i): Using the properties of Stein AG-groupoid, the left invertive law and AG*, $a(bc) = (bc)a = (ac)b = c(ab)$. Hence S is a CA-AG-groupoid.

(iii) and (i) implies (ii): Using the definition of Stein AG-groupoid, the left invertive law and the cyclic associativity we have, $(ab)c = (cb)a = a(cb) = b(ac)$. Hence S is an AG*-groupoid.

Next we provide some counter examples to verify that (i) a Stein AG-groupoid is neither a left commutative nor a right commutative, and (ii) a bi-commutative AG-groupoid is not a Stein AG-groupoid.

Example 8. Table 9 of Example 6 represents a Stein AG-groupoid of order 5. As $(1 \cdot 2)1 \neq (2 \cdot 1)1$, hence it is not left commutative. Also, as $1(1 \cdot 2) \neq 1(2 \cdot 1)$, hence it is also not a right commutative. While Table 11, represents a bi-commutative AG-groupoid of size 3, that is not a Stein AG-groupoid as, $a(aa) \neq (aa)a$.

Table 11. Bi-commutative AG-groupoid that is not Stein AG-groupoid.

·	a	b	c
a	b	b	b
b	c	c	c
c	c	c	c

However, we have the following;

Theorem 10. *A Stein AG-groupoid S is CA, if any of the following hold.*

(i) *S is left commutative.*

(ii) *S is right commutative.*

(iii) *S is bi-commutative.*

Proof. (i) Let S be a left commutative Stein AG-groupoid and $a, b, c \in S$. Then $a(bc) = (bc)a = (cb)a = (ab)c = c(ab)$. Hence S is CA-AG-groupoid.

(ii) Let S be a right commutative Stein AG-groupoid and $a, b, c \in S$. Then $a(bc) = a(cb) = (cb)a = (ab)c = c(ab)$. Hence S is CA-AG-groupoid.

(iii) Obvious.

Lemma 2. *Every Stein CA-AG-groupoid is a semigroup.*

Proof. Let S be a Stein CA-AG-groupoid and $a, b, c \in S$, then $a(bc) = c(ab) = (ab)c$. Hence S is a semigroup.

Stein CA-AG-groupoid

Example 9. Table 12, represent a non-commutative Stein CA-AG-groupoid of order 4, where $3 \cdot 4 \neq 4 \cdot 3$.

Table 12. Non-commutative Stein CA-AG-groupod.

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2

As clear from Example 8 that a Stein AG-groupoid need not to be a bi-commutative AG-groupoid. However, we have the following.

Theorem 11. *Every Stein CA-AG-groupoid is bi-commutative.*

Proof. Let S be a Stein CA-AG-groupoid and $x, y, z \in S$. Then $(xy)z = (zy)x = x(zy) = y(xz) = z(yx) = (yx)z$. Hence S is left commutative. Again, using the given properties we have, $x(yz) = (yz)x = (xz)y = y(xz) = z(yx) = x(zy)$. Thus S is also right commutative. Hence the result follows.

Here we provide a counter example to verify that a bi-commutative CA-AG-groupoid is not necessarily a Stein AG-groupoid.

Example 10. Bi-commutative CA-AG-groupoid of order 4 presented in Table 13, is not a Stein AG-groupoid as $(1 \cdot 1)1 \neq 1(1 \cdot 1)$.

Table 13. Bi-commutative CA-AG-groupoid that is not Stein.

·	1	2	3	4
1	2	3	3	3
2	4	3	3	3
3	3	3	3	3
4	3	3	3	3

Remark 1. Let S be a Stein AG-groupoid, then for all $a, b, c \in S$, $a(bc) = (bc)a = (ac)b = b(ac) \Rightarrow a(bc) = b(ac)$. Thus, every Stein AG-groupoid is an AG**. It is also proved that "every AG** is Bol* [17, Lemma 8] and that each Bol* is paramedial [17, Lemma 9]". Hence every Stein AG-groupoid is paramedial.

Example 11. In Table 9 of Example 6, represent a Stein AG-groupoid, which is not cyclic associative. Table 14 represents an AG-band of order 4, which is not CA as $1(2 \cdot 1) \neq 1(1 \cdot 2)$.

Table 14. AG-band that is not cyclic associative.

$\cdot$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

However, we have the following;

Theorem 12. *A Stein AG-groupoid is CA if it is an AG-band.*

Proof. Let S be a Stein AG-band and $a, b, c \in S$, then using the definition of a Stein AG-groupoid, the left invertive law, definition of AG-band, Remark 1 and the medial law we have,

$$\begin{aligned} a(bc) &= (bc)a = (ac)b = b(ac) = (bb)(ac) \\ &= (cb)(ab) = (ab \cdot b)c = c(ab \cdot b) = c(bb \cdot a) \\ &= c(ba) = (cc)(ba) = (cb)(ca) = (ab)(cc) \\ &= (ab)c = c(ab) \Rightarrow a(bc) = c(ab). \end{aligned}$$

Equivalently, S is a CA-AG-groupoid.

However, a Stein CA-AG-groupoid is not necessarily an AG-band, as clear from the following example.

Example 12. Table 15 represents a Stein CA-AG-groupoid of order 3. As $1.1 \neq 1$, hence it is not an AG-band.

Table 15. Stein AG-groupoid that is not AG-band.

$\cdot$	1	2	3
1	2	1	1
2	1	2	2
3	1	2	2

As every CA-AG-band is commutative semigroup [1, Theorem 2], thus the following corollary is obvious.

Corollary 2. Every CA-AG-band is Stein AG-groupoid.

Now, we discuss role of the (left/right) identity in CA-AG-groupoids. As proved in [6, Theorem 2.3] that "in AG-groupoids the right identity element is always a left identity, while left identity does not imply right identity". Here, we prove that in CA-AG-groupoid the phenomenon is somewhat different, and prove that in CA a left identity becomes the identity, and in this case a CA-AG-groupoid becomes a commutative semigroup.

Lemma 3. *If a CA-AG-groupoid S contains the left identity, then it is also the right identity of S.*

Proof. Let S be a CA-AG-groupoid with the left identity e. Then $ae = e(ae) = e(ea) = ea = a$. Hence e is the right identity.

The following corollary is now obvious.

Corollary 3. In a CA-AG-groupoid S, the following results are equivalent.

(i) e is the left identity of S.

(ii) e is the right identity of S.

(iii) e is the identity of S.

(iv) S is a monoid.

(v) S is commutative.

We provide an example to verify that an AG-groupoid having a left identity is not necessarily a CA-AG-groupoid. In other words, any AG-monoid is not a CA-AG-groupoid.

Example 13. Table 16, represents an AG-monoid of order 3. As $a * (b * c) \neq c * (a * b)$, hence it is not a CA-AG-groupoid.

Table 16. AG-monoid that is not cyclic associative.

$*$	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

However, the following is obvious.

Corollary 4. Every monoid is CA-AG-groupoid.

It has been proved in [7] that locally associative AG-groupoids have associative powers. Here, we characterize CA-AG-groupoid by the powers of its elements.

Lemma 4. *In CA-AG-groupoid S,* $(ab)^2 = (ba)^2\ \forall a, b \in S$.

Proof. Let S be a CA-AG-groupoid, then $\forall a, b \in S$.

$$(ab)^2 = (ab)(ab) = (aa)(bb) = b(aa \cdot b)$$
$$= b(b \cdot aa) = b(a \cdot ba) = (ba)(ba) = (ba)^2.$$

As, by medial law in AG-groupoid S, for all $a, b \in S$,

$$(ab)^2 = (ab)(ab) = (aa)(bb) = a^2b^2.$$

Thus by using this result and Lemma 4, we immediately have that squares of elements commute with each other in CA.

Corollary 5. In CA-AG-groupoid S, $a^2b^2 = b^2a^2$, $\forall a, b \in S$.

Lemma 5. *Let S be a CA-AG-groupoid. Then if for all* x *in S there exist* a *in S such that* (a) $ax = x$ *or* (b) $xa = x$*, then*

(i) $ax^2 = x^2$.

(ii) $x^2a = x^2$.

(iii) $ax^2 = x^2a$.

Proof. (a). (i) By cyclic associativity and the given condition $ax = x$,

$$ax^2 = a(xx) = x(ax) = xx = x^2 \Rightarrow ax^2 = x^2.$$

(ii) By left invertive law and by the given condition

$$x^2a = (xx)a = (ax)x = xx = x^2 \Rightarrow x^2a = x^2.$$

(iii) By (i) and (ii).

(b). (i) By cyclic associativity and given condition $xa = x$

$$ax^2 = a(xx) = x(ax) = x(xa) = xx = x^2 \Rightarrow ax^2 = x^2.$$

(ii) By left invertive law, given condition and cyclic associativity

$$\begin{aligned} x^2a &= (xx)a = (ax)x = (ax)(xa) = a(ax \cdot x) \\ &= x(a \cdot ax) = x(x \cdot aa) = x(a \cdot xa) = x(ax) \\ &= x(xa) = xx = x^2 \Rightarrow x^2a = x^2. \end{aligned}$$

(iii) By (i) and (ii).

Now, we prove that the direct product of two CA-AG-groupoids with same binary operation is cyclic associative and will generalize this idea to two CA-AG-groupoids having arbitrary binary operations.

Theorem 13. *The direct product* $S_1 \times S_2$ *of two CA-AG-groupoids with same binary operation* $(S_1,\cdot)$ *and* $(S_2,\cdot)$ *is a CA-AG-groupoid.*

Proof. Let S_1 and S_2 be two CA-AG-groupoids with same binary operation "·", then $S_1 \times S_2$ is also an AG-groupoid by [13]. To prove that $S_1 \times S_2$ is CA, let (a_1, b_1), (a_2, b_2), $(a_3, b_3) \in S_1 \times S_2$, where $a_1, a_2, a_3 \in S_1$ and $b_1, b_2, b_3 \in S_2$. Then

$$\begin{aligned} (a_1, b_1)\big((a_2, b_2)\,(a_3, b_3)\big) &= (a_1, b_1)\,(a_2a_3,\ b_2b_3) \\ &= (a_1 \cdot a_2a_3,\ b_1 \cdot b_2b_3) = (a_3 \cdot a_1a_2,\ b_3 \cdot b_1b_2) \\ &= (a_3, b_3)(a_1a_2, b_1b_2) = (a_3, b_3)((a_1, b_1)\,(a_2, b_2)) \\ \Rightarrow (a_1, b_1)((a_2, b_2)\,(a_3, b_3)) &= (a_3, b_3)\,((a_1, b_1)\,(a_2, b_2)). \end{aligned}$$

Hence the direct product of two CA-AG-groupoids is cyclic associative.

As proved in [17, Theorem 32] that the direct product of two cancellative AG-groupoids is cancellative. Hence we have the following.

Corollary 6. The direct product $S_1 \times S_2$ of two cancellative CA-AG-groupoids S_1 and S_2 is cancellative CA-AG-groupoid.

Next, we generalize the idea of direct product of CA-AG-groupoids with same binary operation to two arbitrary binary operations and prove that the direct product of any two CA-AG-groupoids is again a CA-AG-groupoid.

Theorem 14. *Let* (S_1, α_1) *and* (S_2, α_2) *be two CA-AG-groupoids with* α_i *binary operations defined on each* S_i *for* $i = 1, 2$. *The direct product of* S_1 *and* S_2 *denoted by* $S = S_1 \times S_2 = \{(a, b)|\ a \in S_1, b \in S_2\}$ *by component wise multiplication on S, then S becomes a CA-AG-groupoid.*

Proof. As (S_1, α_1) and (S_2, α_2) are CA-AG-groupoids with binary operations α_1 and α_2. If $a = (a_1, b_1)$, $b = (a_2, b_2) \in S_1 \times S_2$, where $a_1, a_2 \in S_1$ and $b_1, b_2 \in S_2$, define * on S as follows; $a * b = \{(a_1\alpha_1a_2, b_1\alpha_2b_2)\}$. Clearly, $a * b \in S$. Hence S is a groupoid.

To prove that $S = S_1 \times S_2$ is an AG-groupoid, let $c = (a_3, b_3) \in S_1 \times S_2$, where $a_3 \in S_1$ and $b_3 \in S_2$. Then

$$\begin{aligned} (a * b) * c &= \big((a_1,\ b_1)\,(a_2, b_2)\big)\,(a_3, b_3) \\ &= (a_1\alpha_1a_2,\ b_1\,\alpha_2b_2)\,(a_3,\ b_3) \end{aligned}$$

$$= ((a_1\alpha_1 a_2)\alpha_1 a_3, (b_1\alpha_2 b_2)\alpha_2 b_3)$$
$$= ((a_3\alpha_1 a_2)\alpha_1 a_1,\ (b_3\alpha_2 b_2)\alpha_2 b_1)$$
$$= (a_3\alpha_1 a_2,\ b_3\alpha_2 b_2)\ (a_1, b_1)$$
$$= ((a_3, b_3)\ (a_2, b_2))\ (a_1, b_1)$$
$$\Rightarrow (a * b) * c = (c * b) * a.$$

Hence S is an AG-groupoid. Now to prove that $S = S_1 \times S_2$ is CA, consider

$$a * (b * c) = (a_1, b_1)\ ((a_2, b_2)\ (a_3, b_3))$$
$$= (a_1, b_1)\ (a_2\alpha_1 a_3,\ b_2\alpha_2 b_3)$$
$$= (a_1\alpha_1(a_2\alpha_1 a_3),\ b_1\alpha_2(b_2\alpha_2 b_3))$$
$$= (a_3\alpha_1(a_1\alpha_1 a_2),\ b_3\alpha_2(b_1\alpha_2 b_2))$$
$$= (a_3, b_3)\ (a_1\alpha_1 a_2,\ b_1\alpha_2 b_2)$$
$$= (a_3, b_3)\ ((a_1, b_1)\ (a_2, b_2))$$
$$\Rightarrow a * (b * c) = c * (a * b).$$

Hence S is a CA-AG-groupoid.

4. CONCLUSIONS

We precisely discussed some fundamental characteristics of CA-AG-groupoids and established their relations with some other subclasses of AG-groupoids and with semigroup, monoid etc. We used the modern techniques of GAP, Prover-9 and Mace-4 to produce counterexamples and provide several other examples to improve the standard of this research work.

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