Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 53 (4): 469–475 (2016)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

Characterizing Semirings using Their Quasi and Bi-Ideals

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Abstract: Quasi-ideals in a semiring are the generalization of one-sided right ideals and left ideals. Bi-ideals are generalized form of the quasi-ideals. This paper is concerned with these two types of ideals in the semirings from pure algebraic point of view. We shall characterize three important classes of semirings namely regular semirings, intra-regular semirings and weakly regular semirings by the characteristics of their quasi and bi-ideals along with their right and left-ideals.
Keywords: Quasi-ideal, bi-ideal, regular semiring, intra-regular semiring, weakly-regular semiring

1. INTRODUCTION

Regular rings were introduced by Von Neumann in 1936, in order to clarify certain aspects of Operator Algebras. Since then regular rings have been very extensively studied for their own sake and for the sake of their links with the Operator Algebras. Semirings as generalized rings having no negative elements were initially defined by Vandiver in 1934 [1]. They have wide range applications in theoretical computer science. The algorithms for dynamic programming uses the theory of semirings.

Steinfeld initially defined the quasi-ideals for semigroups and rings respectively in [2] and [3]. Iseki [4] introduced this concept for semirings without zero and proved some results. Shabir et al. [5] characterized semirings by the properties of their Quasi-ideals.

Lajos and Szasz [6] introduced the concept of bi-ideals for the associative rings. Quasi-ideals are the one-sided ideals and bi-ideals are their generalization. In this way, the study of the quasi-ideals and bi-ideals become as important as other ideals.

We present some basic concepts used in the context of semiring theory from the literature for our later pursuit in Section 2. Section 3 deals with the characterization of regular semirings by the properties of their quasi and bi-ideals. The intra-regular and weakly regular semirings are characterized in Sections 4 and 5 respectively. For undefined terms, we refer to [7] and [8].

2. FUNDAMENTAL CONCEPTS

Definition 2.1. A semiring is a nonempty set A possessing two binary operations $+$ (Addition) and $*$ (Multiplication) such that $(A,+)$ is a commutative semigroup and $(A,*)$ is generally a non-commutative semigroup; connecting the two algebraic structures are the distributive laws; $a(b+c)=ab+ac$ and $(a+b)c=ac+bc, \forall a,b,c \in A$.

Definition 2.2. A subsemiring of a semring $(A,+,*)$ is its nonempty subset S provided it is itself a semiring under the operation of A.

Definition 2.3. *A nonempty subset I of a semiring* $(A,+,*)$, is called a right(left) ideal of A if it satisfies the conditions that $x+y \in I$, $x*y \in I \ \forall \ x,y \in A$, and $xa \in I(ax \in I), \forall x \in I, a \in A$. I is called an ideal of A if it is both left and right ideal.

Definition 2.4. Let $(A,+,*)$ be a semiring. A

Received, March 2016; Accepted, November 2016
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quasi-ideal Q of A is a subsemigroup $(Q,+)$ of A such that $AQ \cap QA \subseteq Q$ [4].

Each quasi-ideal of a semiring A is its subsemiring. Every one-sided ideal of A is its quasi-ideal. Since intersection of any family of quasi-ideals of A is its quasi-ideal [5], so intersection of a right ideal R and a left-ideal L of A is a quasi-ideal of A. Both the sum and the product of two or more quasi-ideals of A need not be its quasi-ideal [5].

Definition 2.5. Let $(A,+,*)$ be a semiring. A bi-ideal B is a subsemiring of A if $BAB \subseteq B$.

Every quasi-ideal, product of two quasi-ideals e.g., the product RL of a semiring A is its bi-ideal. However, every bi-ideal is not its quasi-ideal [5]. The product TB and BT of an arbitrary subset T and bi-ideal B of a semiring A are its bi-ideals. Since the product of two bi-ideals of a semiring is a bi-ideal, so is the intersection of their any finite or infinite family.

3. CHARACTERIZING REGULAR SEMIRINGS

Definition 3.1. An element a of a semiring A is called regular if $axa = a$ for some $x \in A$. Semiring A is called regular if each element of A is regular [9]. If a is regular element, then ax and xa are idempotent as $ax.ax = (axa)x = ax$ and $xa.xa = x(axa) = xa$.

We begin to characterize the regular semirings by the following theorem.

Theorem 3.1. *The results given below are equivalents [5]:*

(1) *A is regular,*
(2) $RL = R \cap L$ *for every right-ideal R and left-ideal L of A,*
(3) *(a).* $R^2 = R$, *(b).* $L^2 = L$, *and (c). RL is a quasi-ideal of A,*
(4) *The set of quasi-ideals of A is a regular(multiplicative) semigroup,*
(5) *Each quasi-ideal Q is expressed as* $QAQ = Q$.

Proof. **(1) ⇒ (2):** If R and L are respectively the right and the left ideals of A, then clearly $RL \subseteq R \cap L$. For the converse, let $x \in R \cap L$, then $x \in A$ and as A is regular, so for some $y \in A$, we have $x = xyx = (xy)x \in RL$ because R is right ideal . Thus $R \cap L = RL$.

(2) ⇒ (3): Let $RL = R \cap L$. Since $R \cap L$ is a quasi-ideal [5], RL is a quasi-ideal of A. Now if A is a semiring, then the ideal generated by the right ideal R is $R + AR$, so by (2), we have $R = R \cap (R + AR) = R(R + AR) = R^2 + (RA)R \subseteq R^2 + R^2 \subseteq R^2$: $i.e., R \subseteq R^2$, i.e., $R^2 = R$. Similarly, we can prove that $L^2 = L$.

(3) ⇒ (4): Suppose (3) holds and let K be the set of quasi-ideals of A, then $Q + AQ$ is its left-ideal generated by Q. So by (3), we get $Q \subseteq Q + AQ = (Q + AQ)^2 = (Q + AQ)(Q + AQ) = Q^2 + QAQ + AQQ + AQAQ \subseteq AQ + AQ + AQ + AQ \subseteq AQ$ i.e., $Q \subseteq AQ$. Similarly, we can show that $Q \subseteq QA$. So $Q \subseteq AQ \cap QA$. Since Q is Quasi-ideal, $AQ \cap QA \subseteq Q$ i.e.,

$$AQ \cap QA = Q \quad .... \tag{3.1}$$

Now using 3(c) and (3.1), we get

$$RL = ARL \cap ARL \quad .... \tag{3.2}$$

for right-ideal R and left-ideal L of A. Now we shall prove that Q_1Q_2 of two quasi-ideals Q_1 and Q_2 is a quasi-ideal of A. By property 3(a) and 3(b), we have $AQ_1Q_2 = (AQ_1Q_2)(AQ_1Q_2) = (AQ_1Q_2)(A.AQ_1Q_2)$, and $Q_1Q_2A = (Q_1Q_2A)(Q_1Q_2A) = (Q_1Q_2AA)(Q_1Q_2A)$. Thus using Equation (3.2), we get

$$Q_1Q_2A \cap AQ_1Q_2 = (Q_1Q_2A)(AQ_1Q_2)A \cap A(Q_1Q_2A)(AQ_1Q_2)$$

$$= (Q_1Q_2A)(AQ_1Q_2) \subseteq Q_1(Q_2AQ_2) \subseteq Q_1Q_2$$

i.e., $(Q_1Q_2)A \cap A(Q_1Q_2) \subseteq Q_1Q_2$. So, Q_1Q_2 is a quasi-ideal of A. Since the multiplication of quasi-ideals of the semiring A is associative in K, so K is a semigroup. Finally, we shall show that K is a regular semigroup. If Q is an arbitrary quasi-ideal of A, then the properties 3(a), 3(b) and the Relations (3.1) and (3.2) imply that $Q = QA \cap AQ = (QA.AQ)A \cap A(QA.AQ) = QA.AQ = QAQ \subseteq Q$. So $Q = QAQ$. Thus K is a regular semigroup.

(4) ⇒ (5): Let Q be a quasi-ideal of A. By assumption (4), we can find a Quasi-ideal X of A so that $Q = QXQ \subseteq QAQ \subseteq AQ \cap QA \subseteq Q$; i.e., $Q = QAQ$.

(5) ⇒ (1): Let $b \in A$ and $(b)_L$ and $(b)_R$ be the principal left-ideal and the principal right-ideal of A generated by b, then since the intersection of a left and right ideal is a Quasi-ideal, so $(b)_L \cap (b)_R$ is a quasi-ideal of A, so by (5), we have $(b)_L \cap (b)_R = ((b)_L \cap (b)_R)A((b)_L \cap (b)_R) \subseteq (b)_L A(b)_R$. Since $b \in (b)_L \cap (b)_R$, it follows that $b \in (b)_R A(b)_L$. But $(b)_R A = bA$ and $(b)_L = Ab$, therefore $b \in bA(b)_L = bAb$ i.e., $b \in bAb$ i.e., A is regular.

The following theorem signifies when a bi-ideal of a semiring is a quasi-ideal.

Theorem 3.2. *Let A be a semiring. Then the following assertions hold [5]:*

(1) *Every quasi-ideal Q of A can be written in the form as $Q = R \cap L = RL$, where R is the right and L is the left-ideal,*
(2) *If Q is a quasi-ideal of A, then $Q^2 = Q^3$,*
(3) *Every bi-ideal of A is its quasi-ideal,*
(4) *Every bi-ideal of any two-sided ideal of A is a quasi-ideal of A.*

Proof: **(1)** Let A be a semiring and Q be a quasi-ideal of A then $R=(Q)_r = Q + QA = QA$, and $L = (Q)_l = Q + AQ = AQ$. *Clearly* $Q \subseteq R \cap L = QA \cap AQ \subseteq Q$ i.e., $Q = R \cap L$. But A is a regular semiring, therefore $Q = R \cap L = RL$ by Theorem 3.2.

(2) $Q^3 \subseteq Q^2$ always holds. We have to show that $Q^2 \subseteq Q^3$. By Theorem 3.1, $Q^2$is a quasi-ideal of A. Furthermore $Q^2 = Q^2AQ^2 = QQAQQ \subseteq QQQ = Q^3$i.e., $Q^2 \subseteq Q^3$.

(3) Let B be a bi-ideal of A, then AB respectively BA are left and right-ideal of A, therefore from Theorem 3.1, we have $BA \cap AB = BAAB = BAB \subseteq B$ i.e., $BA \cap AB \subseteq B$ i.e., B is a quasi-ideal of A.

(4) Finally let C be an ideal of A and B be a bi-ideal of C. Then obviously C is a regular subsemiring of A. By **(3)**, B is a quasi-ideal of C. Now $BAB \subseteq BAC \subseteq CAB \subseteq BC \cap CB \subseteq B$ i.e., $BAB \subseteq B$ i.e., B is a bi-ideal of A. Again by **(3)**, B is a bi-ideal of A.

Theorem 3.3. *Let A be a semiring with 1, then A is a division semiring if and only if it has no proper bi-ideals.*

Proof. Let A be a semiring with 1 and $B \neq 0$ be a bi-ideal of A. Let $0 \neq b \in B$. Since A is a division ring, so $bA = Ab = A$. But bA is a right-ideal and A is regular, so $bA \cap Ab = (bA)(Ab)$ [5]. Therefore, Theorem 3.1 gives, $A = bAb \subseteq BAB \subseteq B$ i.e., $A \subseteq B$. But $B \subseteq A$, so $A = B$. Hence A has no proper bi-ideal.

Conversely, if A has no proper bi-ideal. Let $0 \neq a \in A$, then aA and Aa are respectively the right and left-ideal of A and so are bi-ideals. Thus either $aA = A(Aa = A)$ or $aA = 0(Aa = 0)$. But $a \in aA(a \in Aa)$, *so* $aA = A$ and $Aa = A$. Thus A is a division semiring.

Theorem 3.4. *For the semiring A, the following conditions are equivalents:*

(1) *A is regular,*

(2) *For any bi-ideal B of A, $B = BAB$,*

(3) *For any quasi-ideal Q of A, $Q = QAQ$.*

Proof. **(1)** if and only if **(3)** proved in Theorem 3.1. For a regular semiring, the concept of quasi-ideal coincides with the concept of bi-ideal, so **(1)** if and only if **(2).**

Theorem 3.5. *The following results are equivalents for a semiring A for all bi-ideals B, quais-ideals Q and any ideal A:*

(1) *A is regular,*

(2) $I \cap B = BIB$,

(3) $I \cap Q = QIQ$.

Proof: **(1) ⇒ (2):** Suppose A is a semiring and I is any ideal of A and B is any bi-ideal of A then $BIB \subseteq I$ and $BIB \subseteq BAB \subseteq B$. Thus $BIB \subseteq I \cap B$. Let $x \in I \cap B$ then $x = xyx$ for some $y \in A$. Now $x = xyx = x(yxy)x \in BIB$. Thus $I \cap B = BIB$.

(2) ⇒ (3): Let I be any ideal of A and Q any quasi-ideal of A. As every quasi-ideal is bi-ideal, therefore by **(2)**, $I \cap Q = QIQ$.

(3) ⇒ (1): Since A is an ideal of itself, therefore by (3), $A \cap Q = QAQ$, i.e., $Q = QAQ$. Hence by Theorem 3.1, A is a regular semiring.

Theorem 3.6. *The following conditions are equivalent for a semiring A for all its right-ideal R, left-ideal L, Quasi-ideal Q and bi-ideal B:*

(1) *A is regular,*

(2) $R \cap B \subseteq RB$,

(3) $R \cap Q \subseteq RQ$,

(4) $L \cap B \subseteq BL$,

(5) $L \cap Q \subseteq QL$,

(6) $R \cap B \cap L \subseteq RBL$,

(7) $R \cap Q \cap L \subseteq RQL$.

Proof. **(1) ⇒ (2):** Let $x \in R \cap B$, so $x \in R$ & $x \in B$. Since A is regular, there is $y \in A$ such that $x = xyx$. Now $x = (xy)x \in RB$. Thus $R \cap B \subseteq RB$.

(2) ⇒(3): Since every quasi-ideal is a bi-ideals, therefore by **(2),** $R \cap Q \subseteq RQ$.

(3) ⇒ (1): Since every one-sided ideal is a quasi-ideal, therefore by **(3)**, $R \cap L \subseteq RL$. But $RL \subseteq R \cap L$. Therefore, $R \cap L = RL$. Hence by Theorem 3.2, A is a regular semiring. Similarly, we can show that **(1)** if and only if **(4)** if and only if **(5)** if and only if **(1).**

(1) ⇒ (6): $R \cap B \cap L \subseteq (R \cap B) \cap L \subseteq (RB) \cap L$ by **(2).** Now RB is a bi-ideal, so by **(4),** $L \cap (RB) \subseteq (RB)L$. Thus $R \cap B \cap L \subseteq RBL$.

(1)⇒ (7): Since every quasi-ideal is a bi-ideal, therefore by (6), $R \cap Q \cap L \subseteq RQL$ for right-ideal R, left-ideal L and quasi-ideal Q of A.

(7) ⇒ (1): Then by (7), $R \cap A \cap L \subseteq RAL = (RA)L \subseteq RL$. Also $R \cap L = R \cap A \cap L$. *Thus* $R \cap L \subseteq RL$. But $RL \subseteq R \cap L$ always. Hence $R \cap L = RL$. Thus by Theorem 3.1, A is a semiring.

Proposition 3.1. *Let A be a semiring with multiplicative identity 1, then the following are equivalent:*

(1). $R \cap L \subseteq LR$ for any right-ideal R and left-ideal L of A,

(2). Every $a \in A$ can be written as $a = \sum_{i=1}^{n} x_i a^2 y_i$, where $x_i, y_i \in A$.

Proof. (1) ⇒ (2): let $a \in A$. Let $R = aA$ and $L = Aa$ be the right and the left-ideal generated by a respectively. Then by $R \cap L \subseteq LR$, $a \in R \cap L$) $a \in LR$. So $a = \sum_{i=1}^{n} x_i a^2 y_i$, *where* $x_i, y_i \in A$.

(2) ⇒ (1): Let $a \in R \cap L$. Then $a \in R$ and $a \in L$. By (2), $a = \sum_{finite} x_i a^2 y_i = \sum_{finite} (x_i a)(a y_i) \in RL$. So $R \cap L \subseteq LR$.

4. CHARACTERIZING INTRA-REGULAR SEMIRINGS

Intra-regular semiring is also an important class of semirings which can be studied by the properties of their quasi and bi-ideals.

Definition 4.1. A semiring A with multiplicative identity 1 is called intra-regular if every $a \in A$ can be written as $a = \sum_{i=1}^{n} x_i a^2 y_i$, where $x_i, y_i \in A$.

Thus a semiring A with multiplicative identity 1 is called intra- regular if it satisfies one of the conditions of Proposition (3.1).

The next theorem states that the bi-ideals and quasi-ideal are idempotent for intra-regular and regular semiring.

Theorem 4.1. *For a semiring with 1, the following are equivalents:*

(1) *A is both regular and intra-regular,*

(2) *$B^2 = B$ for every bi-ideal B of A,*

(3) *$Q^2 = Q$ for every quasi-ideal Q of A.*

Proof: **(1) ⇒ (2):** Let B be any bi-ideal of A, then $B^2 \subseteq BAB$, since A contains multiplicative identity 1. But $BAB \subseteq B$. Thus $B^2 \subseteq B$. Let $b \in B$ then $b = bxb$ for some $x \in A$, since A is regular. Also since A is intra-regular, $b = \sum_{finite} x_i b^2 y_i$, for some $x_i, y_i \in A$. Thus

$b = bxb = bxbxb = bx(\sum_{finite} x_i b^2 y_i)xb$ $=\sum_{finite}(bxx_i b)\ (by_i xb)$. Since $b \in B$, therefore $b(xx_i)b \in BAB \subseteq B$ and $b(y_i x)b \in BAB \subseteq B$. *Thus* $= \sum_{finite}(bxx_i b)\ (by_i xb) \in BB = B^2$. Hence $B \subseteq B^2$. Consequently $B = B^2$.

(2)⇒ (3): As every quasi-ideal is a bi-ideal, therefore by (2), $Q = Q^2$.

(3) ⇒ (1): Since $R \cap L$ being the intersection of two quasi-ideals is a quasi-ideals, so from **(3),** $R \cap L = (R \cap L)(R \cap L) \subseteq RL$. But $RL \subseteq R \cap L$. *So* $R \cap L = RL$. Now from Theorem (3.1), A is regular. *Now* $R \cap L = (R \cap L)^2 = (R \cap L)(R \cap L) \subseteq LR$. So by Proposition (3.1), A is intra-regular.

Theorem 4.2. *The following conditions are equivalent for a semiring A with*

identity :

(1) *A is both regular and intra-regular,*

(2) $B_1 \cap B_2 \subseteq (B_1B_2) \cap (B_2B_1)$ *for any bi-ideal* B_1 *and* B_2 *of A,*

(3) $B \cap Q \subseteq (BQ) \cap (QB)$ *for any bi-ideal B and quasi-ideal Q of A,*

(4) $Q_1 \cap Q_2 \subseteq (Q_1Q_2) \cap (Q_2Q_1)$ *for any quasi-ideals* Q_1 *and* Q_2 *of A,*

(5) $B \cap R \subseteq (BR) \cap (RB)$ *for any bi-ideal B and right-ideal R of A,*

(6) $Q \cap R \subseteq (QR) \cap (RQ)$ *for any quasi-ideal Q and right-ideal R of A,*

(7) $B \cap L \subseteq (BL) \cap (LB)$ *for any bi-ideal B left-ideal L of A,*

(8) $Q \cap L \subseteq (BQ) \cap (QB)$ *for any quasi-ideal Q and left-ideal L of A,*

(9) $R \cap L \subseteq (LR) \cap (RL)$ *for any right-ideal R and left-ideal R of A,*

Proof. **(1) ⇒ (2):** Since $B_1 \cap B_2$ is a bi-ideal of A, so by Theorem 4.2 , $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1B_2$. Also $B_1 \cap B_2 \subseteq B_2B_1$. Thus $B_1 \cap B_2 \subseteq (B_1B_2) \cap (B_2B_1)$.

(1) ⇒ **(3):** As every quasi-ideal is bi-ideal, therefore by **(2)**, $B \cap Q \subseteq (BQ) \cap (QB)$.

(2) ⇒ **(4):** As every quasi-ideal is bi-ideal, therefore by **(3)**, $Q_1 \cap Q_2 \subseteq (Q_1Q_2) \cap (Q_2Q_1)$

(3) ⇒ **(6):** As every right-ideal is quasi-ideal, therefore by **(4)**, $Q \cap R \subseteq (QR) \cap (QR)$.

(6) ⇒ (9): As every left-ideal is quasi-ideal, therefore by **(6)**, $L \cap R \subseteq (LR) \cap (RL)$.

(9) ⇒ (1): *As* $L \cap R \subseteq (LR) \cap (RL)$, so $R \cap L \subseteq RL$. But $RL \subseteq R \cap L$, therefore $R \cap L = RL$. Hence A is regular. $R \cap L \subseteq LR$ implies that A is intra-regular. Thus we have shown that

(1) ⇒ (2) ⇒ (3) ⇒(4) ⇒ (6) (9) ⇒ (1). Similarly we can show that **(1) ⇒ (2) ⇒ (5) ⇒ (9) ⇒ (1)** and

(1) ⇒ (2) ⇒ (7) ⇒ (8) ⇒ (9) ⇒ (1).

Theorem 4.3. *The following are equivalent for a semiring A with* 1 *for its any right-ideal R, left-ideal L, bi-ideal B and quais-ideal Q:*

(1) *A is both regular and intra-regular,*

(2) $B \cap L \subseteq BLB$,

(3) $B \cap R \subseteq BRB$,

(4) $Q \cap L \subseteq QLQ$,

(5) $Q \cap R \subseteq QRQ$.

Proof. **(1) ⇒ (2):** Take $a \in B \cap L$, then $a \in B$ & $a \in L$. Since A is regular and intra-regular, therefore $a = axa$ and $a = \sum_{finite} x_i a^2 y_i$, where $x, x_i, y_i \in A$. Now $a = axa = axaxa = ax(\sum_{finite} x_i a^2 y_i)xa = a(\sum_{finite} xx_i a^2 y_i xa) = a(\sum_{finite} xx_i a)(a\, y_i xa) \in BLB$, because $ay_i xa \in BAB \subseteq B$.

(2) ⇒ **(4):** Since a quasi-ideal is a bi-ideal, therefore by **(2)**, $Q \cap L \subseteq QLQ$.

(4) ⇒ (1): R being right ideal is a quasi-ideal, so by **(4)**, $R \cap L \subseteq RLR \subseteq LR$. Thus by Proposition (3.1), A is intra-regular. Now let I be any ideal of A, then I is a left-ideal so by **(4)**, $Q \cap I \subseteq QIQ$: On the other hand, $QIQ \subseteq QAQ \subseteq Q$ and $QIQ \subseteq I$, so $QIQ \subseteq Q \cap I$. Thus $Q \cap I \subseteq QIQ$. So by Theorem (3.5), A is regular. Similarly, we can show that **(1) ⇒ (3) ⇒ (5) ⇒ (1).**

5. CHARACTERIZING WEAKLY-REGUALR SEMIRINGS

Analogous to von Neumann regular rings, a ring R is weakly-regular if $x \in (xR)^2$ for each $x \in R$. These rings were introduced by Brown and McCoy [10], later investigated by Rammamurthy [11]. Here we characterize weakly-regular semirings using their quasi-ideals and bi-ideals.

Definition 5.1. A semiring A is called a right weakly-regular semiring if for each $x \in A$, $x \in (xA)^2$. Thus, if A is commutative then A is weakly-regular if and only if A is regular. In general, however, regular semirings form a proper subclass of weakly-regular semirings.

We start to give their characterization by following theorem.

Theorem 5.1. *The following are equivalent for a semiring A with 1:*

(1) *A is weakly-regular,*

(2) $R^2 = R$ *for all right-ideal R of A,*

(3) *For every ideal I of A,* $R \cap I = RI$.

Proof: **(1) ⇒ (2):** Clearly $R^2 \subseteq R$. For the converse, let $x \in R$; so $x \in (xR)^2$. Hence $x \in R^2$, so $R = R^2$.

(2) ⇒ (3): let $x \in I$. Since $x \in (xA) = (xA)^2$, it follows that $x = xy$, for some $y \in I$. For a right-ideal R of A, clearly $RI \subseteq R \cap I$. Let $x \in R \cap I$. Then there exists $y \in I$ such that $x = xy$. Thus $x \in RI$ i.e., $R \cap I \subseteq RI$, so $R \cap I = RI$.

(3) ⇒ (1): Let $x \in A$, Then $x \in (xA) \cap (AxA) = (xA)(AxA) \subseteq (xA^2)(xA) \subseteq (xA)(xA)$ i.e., $x \in (xA)^2$. Hence A is right weakly-regular.

Theorem 5.2. *For a semiring A with identity* 1*, the following conditions are equivalent for all bi-ideal B, quasi-idea Q, ideal I and right-ideal R of A,*

(1) *A is right weakly-regular,*

(2) $B \cap I \cap R \subseteq BI\, R$,

(3) $Q \cap I \cap R \subseteq QI\, R$.

Proof. **(1) ⇒ (2):** Let $x \in \cap I \cap R \Rightarrow x \in B, x \in I$ and $x \in R$. Since $x \in A$ and A is right weakly-regular, therefore $x \in (xA)^2$. i.e., $x = \sum_{finite} xr_i xs_i$ for some, $s_i \in A$. Now, $x = \sum_{finite} xs_i xr_i = \sum_{finite} xs_i (\sum_{finite} xs_i xr_i)\, r_i = \sum_{finite} xa_i xb_i\, xc_i$, where $a_i, a_i, c_i \in A$. Thus $x = \sum_{finite} x(a_i xb_i\, x)c_i \in BIR$. Hence $B \cap I \cap R \subseteq BIR$.

(1) ⇒ (3): Every quasi-ideal is a bi-ideal, therefore by **(2)**, $Q \cap I \cap R \subseteq QIR$.

(3) ⇒ (1): Taking $Q = R$, $I = I$ and $R = A$, we get from **(3)**, $R \cap I \cap A \subseteq RIA$. As $R \cap I \cap A = RIA$, and $RIA \subseteq RI$. Thus $R \cap I \subseteq RI$. But $RI \subseteq R \cap I$, so by Theorem (5.1), A is right weakly-regular.

Theorem 5.3. *For a semiring A with* identity*, the following are equivalent for all bi-ideals B and two-sided ideals I:*

(1) *A is right weakly-regular,*
(2) $B \cap I \subseteq BI$,
(3) $Q \cap I \subseteq QI$.

Proof. **(1) ⇒ (2):** Let $x \in B \cap I$, then $x \in B$ and $x \in I$. Since $x \in A$ and A is right weakly-regular, therefore $x \in (xA)^2$ i.e., $x = \sum_{finite} xr_i xs_i$ for some r_i, $s_i \in A$. Now $x = \sum_{finite} x(r_i xs_i) \in BI$. Thus $B \cap I \subseteq BI$.

(2) ⇒ (3): Since a quasi-ideal is a bi-ideal, therefore by **(2)**, $Q \cap I \subseteq QI$.

(3) ⇒ (1): Since a one-sided ideal is a quasi-ideal , therefore by **(3)**, $R \cap I \subseteq RI$. But $RI \subseteq R \cap I$. Therefore $R \cap I = RI$. Hence by Theorem (5.1), A is a right weakly-regular.

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