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**Research Article** 

# **On Strongly \*-Graphs**

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Abstract: A graph G = (V, E) is said to be strongly \*-graph if there exists a bijection  $f : V \to \{1, 2, ..., n\}$  in such a way that when an edge, whose vertices are labeled *i* and *j*, is labeled with the value i + j + ij, all edge labels are distinct. In this paper we get an upper bound for the number of edges of any graph with *n* vertices to be strongly\*- graph, and we make an algorithm to check any graph if it is a strongly\*- graph or not. Also, we study some new families to be strongly\*- graphs.

**Keywords:** Strongly \*-graph / labeling, C<sup>++</sup> programming Language

Classification Code: 05C78

#### 1. INTRODUCTION

By a graph *G* we mean a finite, undirected, connected graph without loops or multiple edges. We denote by *n* and *m* the order and size of the graph *G* respectively. Terms not defined here are used in the sense of Harary [1]. A variation of strong multiplicity of graphs is a strongly \*-graph. A graph of order *n* is said to be a strongly \*-graph if its vertices can be assigned the values 1, 2, ..., n in such a way that, when an edge is labeled with the value i + j + ij, where *i* and *j* are the labels of its vertices, all edge labels are distinct [2]. Adiga and Somashekara [3] gave a strongly \*-labeling for all trees, cycles, and grids. They further consider the problem of determining the maximum number of edges in any strongly \*-graph of a given order and relate it to the corresponding problem for strongly multiplicative graphs. Seoud and Mahran [4-5] give some technical necessary conditions for a graph to be strongly \*-graph.

Babujee and Vishnupriya [6] have proved the following families to be strongly \*-graphs:  $C_n \times P_2$ ;  $(P_2 \cup \overline{K_m}) + \overline{K_2}$ , windmills  $K_3^{(n)}$ , and jelly fish graphs J(m, n) obtained from a 4-cycle  $v_1, v_2, v_3, v_4$  by joining  $v_1$  and  $v_3$  with an edge and appending m pendent edges to  $v_2$  and n pendent edges to  $v_4$ . Babujee and Beaula [7] gave a strongly \*-labelings for  $C_n$  and  $K_{n,m}$ . Babujee, Kannan, and Vishnupriya [8] gave a strongly \*-labelings for  $W_n$ ,  $P_n$ , fans, crowns,  $(P_2 \cup mK_1) + \overline{K_2}$ , and umbrellas. An example for a strongly \*-graph with p = 6, q = 11 is shown in Fig. 1 while  $K_6$  is not a strongly \*-graph as shown in Fig. 2.

### 2. NEW GENERAL RESULTS

In this section we make an algorithm which find all possible edge labels; under the condition of strongly \*-labeling; for any graph of order n and print out the edge labels which are repeated and their corresponding adjacent vertices in case this graph is a complete graph. An example is shown in the Fig. 3. Then counting these repeated edges and subtracting them from the number of edges of a complete graph gives an upper bound for the number of edges of any graph of order n to be strongly \*-graph.

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Fig. 1





Fig. 3



Fig. 4

Table 1 shows the upper bound of the number of edges of a graph of order *n* to be strongly \*-graph up to n = 70.

order	repetitions in edges labels	Upper bound for the no. of edges	order	repetitions in edges labels	Upper bound for the no. of edges
3	0	3	37	223	443
4	0	6	38	240	463
5	1	9	39	263	478
6	1	14	40	263	517
7	2	19	41	287	533
8	3	25	42	287	574
9	6	30	43	309	594
10	6	39	44	332	614
11	11	44	45	353	637
12	11	55	46	353	682
13	16	62	47	382	699
14	21	70	48	395	733
15	26	79	49	421	755
16	26	94	50	444	781
17	34	102	51	471	804
18	34	119	52	471	855
19	42	129	53	502	876
20	50	140	54	525	906
21	59	151	55	558	927
22	59	172	56	584	956
23	72	181	57	611	985
24	78	198	58	611	1042
25	89	211	59	651	1060
26	98	227	60	651	1119
27	111	240	61	680	1150
28	111	267	62	714	1177
29	128	278	63	748	1205
30	128	307	64	776	1240
31	142	323	65	815	1265
32	156	340	66	815	1330
33	171	357	67	851	1360
34	185	376	68	883	1395
35	206	389	69	926	1420
36	206	424	70	926	1489

**Table 1.** An upper bound for the number of edges of a graph with a given order to be strongly \*-graph.

**Remark:** It's clear that all complete graphs  $K_n$ ,  $n \ge 5$  are not strongly \*-graphs for the same reason in case of  $K_6$ .

From this upper bound we find that all graphs of order  $\leq 6$  are strongly \*-graphs except  $K_5$  and  $K_6$ , and this can be easily proved as the two edges joining the vertices 2,3 and 1,5 in both graphs  $K_5$  and  $K_6$ have the same edge label which is equal 11, then this make the graphs  $K_5$  and  $K_6$  non-strongly \*-graph. So for any graph of order  $\leq 6$  either join the two vertices labeled 1 and 5 or 2 and 3 but not both regardless for the number of edges in this graph. Moreover, we give an algorithm to check any (n, m)graph whether it is strongly \*-graph or not, and also give all possible labelings for this graph using the labeling function  $f: V(G) \rightarrow \{1, 2, ..., n\}$  as follows:

Given the number of vertices *n*, the number of edges *m* and the vertices adjacent to each edge:

- INPUT: The number of vertices *n*, the number of edges *m* of the graph.
- OUTPUT: State whether the graph is strongly \*-graph or not and display the vertex labelings if it is strongly \*-graph.
- Step 1: Set v; (array with length n stores the labels of the vertices)

adj1; (array with length *m* stores the labels of the first vertex adjacent to each edge)

adj2; (array with length m stores the labels of the second vertex adjacent to each edge) edgelabel; (array with length m stores the calculated labels of all edges)

- Step 2: Enter the adjacent vertices to each edge (adj1 and adj2);
- Step 3: Initialize v = [1 2 ... n];

x = 0; (used to count the number of possible labelings of all the permutations of the vector v)

- Step 4: Initialize FLAG = 0; (used to decide whether to display v or not)
- Step 5: Calculate the label of each edge : edgelabel = v[adj1] + v[adj2] + v[adj1] \* v[adj2];
- Step 6: FOR index i = 1: m 1 (check whether the graph is strongly \*-graph or not)
- Step 7: FOR index j = i + 1: m
- Step 8: If *edgelabel* [i] = edgelabel [j] then

Set FLAG = 1;

Permute v; (make another permutation of the vector v)

Go to step 4

Step 9: If FLAG = 0 then OUTPUT (v); (display the labels of the vertices) x = x + 1;

Step 10: permute v; (make another permutation of the vector v)

Step 11: Go to step 4

- Step 12: If x = 0 then OUTPUT (No Strongly \*-Labelings for this Graph);
- Step 13: STOP;

We implement this algorithm using  $C^{++}$  programming language.

An example for a graph with n = 9 and m = 20 is shown in Fig. 4 and Fig. 5 (we use the 1<sup>st</sup> labeling given by the program).

C:\Users\Abou Shady\Desktop\Strongly Star-graph.exe						
Enter the number of Vertices = 9						
Tates surbay of advance 20						
Enter number of eages - 20	Enter the adjacent vertices to edge no. 12 : 1 5					
Enter the adjacent vertices to edge no. 1 : 1 2	- Enter the adjacent wentices to edge no. 13 · 1					
Enter the adjacent vertices to edge no. $2:2$	6					
3	Enter the adjacent vertices to edge no. 14 : 1					
Enter the adjacent vertices to edge no. 3 : 3	7					
4	Enter the adjacent vertices to edge no. 15 : 1 $\mathfrak{g}$					
Enter the adjacent vertices to edge no. 4 : 4 5						
	Enter the adjacent vertices to edge no. 16 = 2 5					
6	Enter the adjacent vertices to edge no. $17:2$					
Enter the adjacent vertices to edge no. 6 : 6	7					
7	Enter the adjacent vertices to edge no. 18 : 2					
Enter the adjacent vertices to edge no. 7 : 7	у — — — — — — — — — — — — — — — — — — —					
	Enter the adjacent vertices to edge no. 19 : 4 8					
Enter the adjacent vertices to edge no. 8 : 8 9	Fotew the adjacent vewtices to edge no 20 : 5					
Enter the adjacent vertices to edge no. 9 : 9 1	7 					
Enter the adjacent vertices to edge no. 10 : 1 3	Strongly <del>×-</del> Labelings are :					
Enter the adjacent vertices to edge no. 11 : 1 4	(1 2 4 5 6 3 7 9 8 )					





#### 3. SOME NEW FAMILIES OF GRAPHS THAT ARE STRONGLY \*-GRAPHS

In this section we introduce some new families that are found to be strongly \*-graphs such as the one point union of *m* copies of the complete bipartite graph  $K_{2,n}^{(m)}$ ,  $C_n \circ K_{1,m}$ , the graph obtained from  $F_n$  by inserting one vertex between every two consecutive vertices of  $P_n$ ,  $P_n \odot \overline{K}_m$ , the triangular snake  $T_n$ ,  $T_n \odot K_1$ , The Sun Flower SF(n),  $S_m \cup S_n$ ,  $B_{m,n}$ ,  $P_n \times C_4$  and  $P_n \wedge P_m$ .

**Theorem 3.1:** The one point union of *m* copies of the complete bipartite graph  $K_{2,n}^{(m)}$  is strongly \*-graph.

**Proof:** The graph  $K_{2,n}^{(m)}$  has |V| = nm + m + 1 vertices and |E| = 2nm edges. Let the set of vertices be as follows:  $\{v_0; v_1, v_1^1, v_2^1, \dots, v_n^1; v_2, v_1^2, v_2^2, \dots, v_n^2; \dots; v_m, v_1^m, \dots, v_n^m\}$  as described in Fig. 6.

We define the labeling function  $f : V \rightarrow \{1, 2, \dots, nm + m + 1\}$  as follows:

$$\begin{split} f(v_0) &= 1 \\ f(v_j) &= mn + j + 1 ; & 1 \le j \le m , \\ f(v_i^j) &= (j-1)n + i + 1 ; & 1 \le i \le n , \ 1 \le j \le m . \end{split}$$

*Example 3.1:*  $K_{2,4}^{(3)}$  is a strongly \*-graph as shown in Fig. 7.

**Theorem 3.2:** The graph  $C_n \circ K_{1,m}$  (obtained by identifying a vertex of  $C_n$  with the centre of  $K_{1,m}$ ) is a strongly \*-graph.

**Proof:** The graph  $C_n \circ K_{1,m}$  has |V| = n + m vertices and |E| = n + m edges. Let the set of vertices be as follow:  $V(C_n \circ K_{1,m}) = \{u_1, u_2, \dots, u_m; v_1, v_2, \dots, v_n\}$  as described in Fig. 8.

Since  $\forall n, m, k, s \in N$ , if nm + n + m = ks + k + s : n < k then n < k < s < m.

*Example 3.2:* for  $n = 1, m = 5, k = 2, s = 3 \Longrightarrow nm = ks = 11$ .

Then we define the labeling function  $f : V \rightarrow \{1, 2, ..., n + m\}$  as follows:

$$f(v_i) = \begin{cases} n - 2i + 2, & 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 2i - n - 1, & \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \\ f(u_j) = j + n; & 1 \le j \le m \end{cases}$$

Note that  $f(u_i) > f(v_1) > f(v_i), \forall i, j$ , then all the edges' labels must be distinct.

*Example 3.3:*  $C_7 \circ K_{1,4}$  and  $C_8 \circ K_{1,4}$  are a strongly \*-graphs as shown in Fig. 9(a) and Fig. 9(b) respectively.

**Theorem 3.3:** The graph obtained from  $F_n$  by inserting one vertex between every two consecutive vertices of  $P_n$  is a strongly \*-graph.

**Proof:** This graph has |V| = 2n vertices and |E| = 3n - 2 edges. Let the set of vertices be as follow:  $\{u_0, v_1, v_2, \dots, v_{2n-1}\}$  as described in Fig. 10.

We define the following function to label this graph

$$f(u_0)=2n\,,$$

 $f(v_{2i}) = i, \quad 1 \le i \le n - 1$ ,

$$f(v_{2i-1}) = n + i - 1, \quad 1 \le i \le n$$
.

Using this labeling function, the edge labels are all distinct and in an ascending order.

*Example 3.4:* The graph obtained by inserting in  $F_5$  one vertex between every two consecutive vertices of  $P_5$  is a strongly \*-graph as shown in Fig. 11.













Fig. 9(a)

Fig. 9(b)







Fig. 11



Fig. 12



**Theorem 3.4:** The corona  $P_n \odot \overline{K}_m$  is a strongly \*-graph.

**Proof:** This graph has the set of vertices  $V(P_n) = \{v_1, v_2, ..., v_n\}$  and  $V(\overline{K_m}) = \{v_1^1, v_1^2, ..., v_1^m; v_2^1, v_2^2, ..., v_2^m; ...; v_n^1, v_n^2, ..., v_n^m\}$  with total number of vertices |V| = n(m+1) and total number of edges |E| = nm + (n-1) as shown in Fig. 12.

We will use the following labeling function:

$$f(v_i) = (i - 1)(m + 1) + 1, \ 1 \le i \le n ,$$
  
$$f(v_i^j) = f(v_i) + j, \ 1 \le j \le m.$$

Using this labeling function, the edge labels are all distinct and in an ascending order.

*Example 3.5:* The corona  $P_4 \odot \overline{K}_4$  is a strongly \*-graph as shown in Fig. 13.

**Definition:** The triangular snake  $T_n$  is the graph which is obtained from a path  $P_n$  with vertices  $\{v_1, v_2, ..., v_n\}$  by joining the vertices  $v_i$  and  $v_{i+1}$  to a new vertex  $u_i$  for i = 1, 2, ..., n - 1 as shown in Fig. 14.

**Theorem 3.5:** The triangular snake  $T_n$  is a strongly \*-graph.

**Proof:** This graph has the set of vertices  $V = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-1}\}$  with total number of vertices |V| = 2n - 1 and total number of edges |E| = 3(n - 1).

We will use the following labeling function:

$$f(v_i) = 2i - 1, \ 1 \le i \le n,$$
  
 $f(u_j) = 2j, \ 1 \le j \le n - 1.$ 

Using this labeling function, the edge labels are all distinct and in an ascending order.

*Example 3.6:* The triangular snake  $T_6$  is a strongly \*-graph as shown in Fig. 15.

**Theorem 3.6:** The corona  $T_n \odot K_1$  is a strongly \*-graph.

**Proof:** This graph has the set of vertices  $\{v_1, v_2, \dots, v_n; u_1, u_2, \dots, u_{n-1}; a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_{n-1}\}$  as shown in Fig. 16 with total number of vertices |V| = 4n - 2 and total number of edges |E| = 5n - 4.

We will use the following labeling function:

 $f(v_1) = 2, f(a_i) = 4(i-1) + 1, \ 1 \le i \le n,$   $f(v_i) = 4(i-1) - 1, \ 2 \le i \le n, f(b_j) = 4j + 2, \ 1 \le j \le n - 1.$  $f(u_j) = 4j, \ 1 \le j \le n - 1$ 

Using this labeling function, edge labels are all distinct and in an ascending order.

We have  $v_i$  is adjacent to  $v_{i+1}$  and  $b_{i-1}$  is adjacent to  $u_{i-1}$ , and  $f(v_i) < f(u_{i-1}) < f(b_{i-1}) < f(v_{i+1})$ . Then, the label of the edge incident to the vertices  $v_i$  and  $v_{i+1}$  may be equal to that of the edge incident to the vertices  $b_{i-1}$  and  $u_{i-1}$ , but using the previous labeling function, let  $f(u_{i-1}) = x$  it follows that  $f(v_i) = x - 1$ ,  $f(v_{i+1}) = x + 3$  and  $f(b_{i-1}) = x + 2$  as shown in Fig. 17, then the label of the two described edges will be equal if and only if x + (x + 2) + x \* (x + 2) = (x - 1) + (x + 3) + (x - 1) \* (x + 3), i.e.  $0 = -3 \Rightarrow$  contradiction.

*Example 3.7:* The corona  $T_5 \odot K_1$  is a strongly \*-graph as shown in Fig. 18.

**Definition:** The Sun Flower SF(n) was defined by Lee and Seah [9] as the graph obtained from the cycle  $C_n$  with vertices  $\{v_1, v_2, ..., v_n\}$  and new vertices  $\{u_1, u_2, ..., u_n\}$  such that  $u_i, i = 1, 2, ..., n - 1$ , is connected to  $v_i$  and  $v_{i+1}$ , and  $u_n$  is connected to  $v_n$  and  $v_1$  as shown in Fig. 19.

**Theorem 3.7:** The Sun Flower SF(n) is a strongly \*-graph.



Fig. 14













Fig. 18







Fig. 20

**Proof:** This graph has 2n vertices and 3n edges, the set of vertices are  $\{v_1, v_2, ..., v_n; u_1, u_2, ..., u_n\}$ . The labeling function  $f: V(SF(n)) \rightarrow \{1, 2, ..., n\}$  is defined as follows:

$$f(v_i) = \begin{cases} 4i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 4(n-i)+1, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n \end{cases}$$
$$f(u_i) = \begin{cases} 4i-2, & 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 4(n-i)+3, & \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases}$$

Note that if  $v_i$  and  $u_j$  are adjacent, then  $|f(v_i) - f(u_j)| \le 2$  and if  $v_i$  and  $v_j$  are adjacent, then  $|f(v_i) - f(v_j)| \le 4$ . Then there will exist 2 cases as follows:

*Case I:* The vertices  $v_i$  and  $u_i$  are adjacent and the vertices  $v_{n-i+1}$  and  $v_{n-i}$  are adjacent, such that  $f(v_{n-i+1}) < f(u_i) < f(v_i) < f(v_{n-i}), \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ , so the labels of these two edges may be equal as shown in Fig. 20.

Now, assume  $f(u_i) = x \Rightarrow f(v_i) = x + 2$ ,  $f(v_{n-i}) = x + 3$  and  $f(v_{n-i+1}) = x - 1$ , then the labels of these two edges will be equal if and only if x + (x + 2) + x \* (x + 2) = (x - 1) + (x + 3) + (x - 1) \* (x + 3), i.e. 0 = -3 which gives a contradiction.

*Case II:* The vertices  $v_{n-i}$  and  $u_{n-i}$  are adjacent and the vertices  $v_i$  and  $v_{i+1}$  are adjacent, such that  $f(v_i) < f(v_{n-i}) < f(u_{n-i}) < f(v_{i+1}), \ 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ , so the labels of these two edges may be equal as shown in Fig. 21.

Now, assume  $f(v_{n-i}) = x \Longrightarrow f(u_{n-i}) = x + 2$ ,  $f(v_{i+1}) = x + 3$  and  $f(v_i) = x - 1$ , then the labels of these two edges will be equal if and only if x + (x + 2) + x \* (x + 2) = (x - 1) + (x + 3) + (x - 1) \* (x + 3), i.e. 0 = -3 which gives a contradiction.

*Example 3.8: SF*(7) and *SF*(8) are strongly \*-graphs as shown in Fig. 22.

**Theorem 3.8:** The Union  $S_m \cup S_n$  is a strongly \*-graph.

**Proof:** The stars  $S_m$  and  $S_n$  has the set of vertices  $V(S_m) = \{w_1, v_1, v_2, ..., v_m\}$  and  $V(S_n) = \{w_2, u_1, u_2, ..., u_n\}$  where  $w_1$  and  $w_2$  are the centers of  $S_m$  and  $S_n$  respectively as shown in Fig. 23 with total number of vertices |V| = m + n + 2 and total number of edges |E| = m + n.

We define the following labeling function:

$$f(v_i) = i, \ 1 \le i \le m, \qquad f(w_2) = m + n + 2.$$
  

$$f(u_j) = m + j, \ 1 \le j \le n,$$
  

$$f(w_1) = m + n + 1,$$

*Example 3.9:* The Union  $S_9 \cup S_{10}$  is a strongly \*-graphs as shown in Fig. 24.

**Theorem 3.9:** The Bistar  $B_{m,n}$ , the graph obtained by joining the centers of two stars with an edge, is a strongly \*-graph.

**Proof:** Let the stars be  $S_m$  and  $S_n$ . Then this graph has the set of vertices  $V(S_m) = \{w_1, v_1, v_2, ..., v_m\}$  and  $V(S_n) = \{w_2, u_1, u_2, ..., u_n\}$  where  $w_1$  and  $w_2$  are the centers of  $S_m$  and  $S_n$  respectively as shown in Fig. 25 with total number of vertices |V| = m + n + 2 and total number of edges |E| = m + n + 1.









Fig. 22







We define the following labeling function:

 $f(v_i) = i, \ 1 \le i \le m, \qquad f(w_2) = m + n + 2.$  $f(u_j) = m + j, \ 1 \le j \le n, \qquad f(w_1) = m + n + 1,$ 

*Example 3.10:*  $B_{9,10}$  is a strongly \*-graphs as shown in Fig. 26.

**Theorem 3.10:** The Cartesian Product  $P_n \times C_4$  is a strongly \*-graph.

**Proof:** This graph has |V(G)| = 4n vertices and |E(G)| = 4(2n-1) edges. Let the set of vertices  $V(P_n \times C_4) = \{v_i^j : 1 \le i \le n, 1 \le j \le 4\}$  as shown in Fig. 27.

We define the following labeling function:

 $f(v_i^j) = i + 4(j-1), \ 1 \le i \le 4, 1 \le j \le n.$ 

Using this labeling we note that for any two adjacent vertices  $v_i^j$  and  $v_r^s |f(v_i^j) - f(v_r^s)| \le 4$ . We will find three different cases at which there exist two adjacent vertices with labels x and y and another two adjacent vertices with labels z and w such that  $x \le z \le w \le y$ , so we will assume these four labels as follows:

*Case I:* The vertices labeled x and x + 4 are adjacent and the vertices labeled x + 1 and x + 3 are adjacent as shown in Fig. 28(a), these two edges will have equal labels if and only if x + x + 4 + x(x + 4) = x + 1 + x + 3 + (x + 1)(x + 3), i.e  $0 = 3 \implies$  contradiction.

*Case II:* The vertices labeled x and x + 4 are adjacent and the vertices labeled x + 2 and x + 3 are adjacent as shown in Fig. 28(b), these two edges will have equal labels if and only if x + x + 4 + x(x + 4) = x + 2 + x + 3 + (x + 2)(x + 3), i.e.  $x = -7 \implies$  contradiction.

*Case III:* The vertices labeled x and x + 4 are adjacent and the vertices labeled x + 1 and x + 2 are adjacent as shown in Fig. 28(c), these two edges will have equal labels if and only if x + x + 4 + x(x + 4) = x + 1 + x + 2 + (x + 1)(x + 2), i.e x = 1, but the vertices labeled 2 and 3 are not adjacent  $\Rightarrow$  contradiction.

*Example 3.11:*  $P_4 \times C_4$  is a strongly \*-graphs as shown in Fig. 29.

**Theorem 3.11:** The conjunction  $P_n \wedge P_m$  is a strongly \*-graph.

**Proof:** This graph has the set of vertices  $V(G) = \{v_1^1, v_2^1, \dots, v_m^1; v_1^2, v_2^2, \dots, v_m^2; \dots; v_1^n, v_2^n, \dots, v_m^n\}$  with total number of vertices |V(G)| = nm and total number of edges |E(G)| = 2(n-1)(m-1) as shown in Fig. 30.

Using the labeling function  $f: V(G) \rightarrow (1,2,3,...,n)$  as follows:

$$\begin{split} f\left(v_{2j-1}^{2i-1}\right) &= j + m(i-1), \qquad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \qquad 1 \le j \le \left\lceil \frac{m}{2} \right\rceil \\ f\left(v_{2j}^{2i}\right) &= \left\lceil \frac{m}{2} \right\rceil + j + m(i-1), \qquad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \qquad 1 \le j \le \left\lceil \frac{m}{2} \right\rceil \\ f\left(v_{2j}^{2i-1}\right) &= \left\lceil \frac{nm}{2} \right\rceil + j + m(i-1), \qquad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \qquad 1 \le j \le \left\lceil \frac{m}{2} \right\rceil \\ f\left(v_{2j-1}^{2i}\right) &= \left\lceil \frac{nm}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor + j + m(i-1), \qquad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \qquad 1 \le j \le \left\lceil \frac{m}{2} \right\rceil \end{split}$$

We notice that this labeling function makes all the edge labels distinct.

*Example 3.12:*  $P_4 \wedge P_5$  and  $P_5 \wedge P_6$  are strongly \*-graphs as shown in Fig. 31, respectively.





Fig. 24



Fig. 25







Fig. 27







(c)

(a)

Fig. 28

**(b)** 



Fig. 29



Fig. 30





Fig. 31

#### 4. **REFERENCES**

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