Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 54 (2): 197–205 (2017)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

m-Exponential Convexity of Refinements of Hermite-Hadamards Inequality

Rishi Naeem*

School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan

Abstract. This paper narrates m-exponential convexity and log-convexity. For this investigation positive functionals are applied which associate with the refinement of Hermite Hadamard inequality (cited from [1]). With the results that are obtained, some families of functions related to them are presented. To construct means with Stolarsky property, Lagrange and Cauchy type mean value theorems are also given.

Keywords and Phrases: Convex function, Hermite Hadamard inequality, log-convexity, m-exponential convexity

2010 *Mathematics Subject Classification.* Primary 26A24; 26A48; 26A51; 26D15

1. PRELIMINARIES

An important and useful inequality in literature is known as Hadamard inequality stated as follows: Let $\psi: S = [\alpha, \beta] \to \mathbb{R}$ be a convex function, then

$$\psi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)\, dr \leq \frac{\psi(\alpha)+\psi(\beta)}{2}.$$

By considering $r = \frac{\alpha+\beta}{2}$ with its two convex combinations, the refinement of Hadamard inequality is obtained in [1].

Theorem 1.1. [1] Other notable literature about inequalities are [2-5]. *Consider a closed real interval* $S = [\alpha, \beta]$, $\gamma, \delta \in S$ *and let* $\psi: S \to \mathbb{R}$ *is a convex function. Suppose*

(1) $$a = \frac{\gamma-\alpha}{\beta-\alpha}, \quad b = \frac{\beta-\gamma}{\beta-\alpha}, \quad c = \frac{\delta-\alpha}{\beta-\alpha}, \quad d = \frac{\beta-\delta}{\beta-\alpha}.$$

Then

(2) $$\psi\left(\frac{\alpha+\beta}{2}\right) \leq a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\gamma+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)\, dr \leq \frac{1}{2}[c\psi(\alpha) + d\psi(\beta) + \psi(\delta)] \leq \frac{\psi(\alpha)+\psi(\beta)}{2}.$$

Remark 1.2. *The functional forms of above refinement are given as:*

Received, August 2016; Accepted, May 2017
*E-mail: rishi.naeem@sns.nust.edu.pk

(3) $$\Gamma_1(\psi) = a\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\gamma+\beta}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right).$$

(4) $$\Gamma_2(\psi) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)\,dr - a\psi\left(\frac{\alpha+\gamma}{2}\right) - b\psi\left(\frac{\gamma+\beta}{2}\right).$$

(5) $$\Gamma_3(\psi) = \frac{1}{2}[c\psi(\alpha) + d\psi(\beta) + \psi(\delta)] - \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)\,dr\,.$$

(6) $$\Gamma_4(\psi) = (1-c)\psi(\alpha) + (1-d)\psi(\beta) - \psi(\delta)\,.$$

Then for $1 \leq j \leq 4$, $\Gamma_j(\psi)$ *are positive.*

J. Pečarić and J. Perić [6] introduced the notion of m-exponential convexity. We used an effective technique from [7] to construct m-exponentially convex functions. Some notable results related to exponential convexity are investigated in [8-20].

m-exponential convexity of the functions related with the refinement of Hermite Hadamard inequality (2) is examined in section 2. In addition, the results about exponential and log-convexity are deduced. To construct means with Stolarsky property, Lagrange and Cauchy type mean value theorems are also given.

2. EXPONENTIAL CONVEXITY

If a function $\psi: S \to \mathbb{R}$ of real values is continuous and m-exponentially J-convex on S then it is m-exponentially convex; hence it is an exponentially convex function (details given in [6]).

Remark 2.1. *Let* $\psi: S \to \mathbb{R}$ is a *positive function of real values. The function* ψ *is log-*J*-convex if and only if it is 2-exponentially* J *- convex. If* ψ *is continuous, then converse holds.*

Remark 2.2. *The function* ψ *is increasing on its domain if provided that for all* s_1, s_2 *in* S *the divided difference* $[s_1, s_2; \psi]$ *is non negative.*

A very basic and useful inequality for log-convex functions is given in the lemma mentioned below.

Lemma 2.3. [13]: *If for* $k, s, t \in S$ *with* $k < s < t$ *the function* $\psi: I \to \mathbb{R}$ *is log-convex, then,*

$$\left(\psi(s)\right)^{t-k} \leq \left(\psi(k)\right)^{t-s}\left(\psi(t)\right)^{s-k}.$$

Now we investigate m-exponential and exponential convexity by applying the functionals on a given family of functions. For the construction of exponentially convex functions, we apply the technique from [8].

Theorem 2.4. *Consider a family of functions on* $S \subseteq \mathbb{R}$ *defined by* $\Xi = \{h_r | r \in I \subset \mathbb{R}\}$*, such that for every three different points* $s_1, s_2, s_3 \in S$ *the function* $r \mapsto [s_1, s_2, s_3;\, h_r]$ *on the interval* I *is* m*-exponentially* J*-convex. Take* $\Gamma_1(\psi)$ *into account which is mentioned in Remark 1.2. Then the* m*-exponential* J*-convexity of* $r \mapsto \Gamma_1(h_r)$ *holds on* I*. The function* $r \mapsto \Gamma_1(h_r)$ *is* m*-exponentially convex on* I *if it is continuous also.*

Proof. Suppose r_j, r_k be the elements of I, $r_{jk} = \frac{r_j + r_k}{2}$ and c_j, c_k are real numbers for $j, k = 1, \cdots, m$. Consider the function Ω on S defined as

$$\Omega(s) = \sum_{j,k=1}^{m} c_j c_k h_{r_{jk}}(s).$$

Thus Ω is continuous, since it is the linear combination of continuous functions. According to the hypothesis of function $r \mapsto [s_1, s_2, s_3;\ h_r]$ is m-exponentially J-convex, yields

$$[s_1, s_2, s_3; \Omega] = \sum_{j,k=1}^{m} c_j c_k \left[s_1, s_2, s_3; h_{r_{jk}}\right] \geq 0.$$

This implies the convexity of Ω on S. Thus, we get $\Gamma_1(\Omega)$ is non negative. By the linear property of Γ_1 we have

$$\sum_{j,k=1}^{m} c_j c_k \Gamma_1\left(h_{r_{jk}}\right) \geq 0,$$

concluding the m-exponential J-convexity of function $r \mapsto \Gamma_1(h_r)$ on I.

The above model results the following outcomes.

Corollary 2.5. *Consider a family of functions on $S \subseteq \mathbb{R}$ defined by $\Xi = \{h_r | r \in I \subset \mathbb{R}\}$, such that for every three different points $s_1, s_2, s_3 \in S$ the function $r \mapsto [s_1, s_2, s_3;\ h_r]$ on the interval I is exponentially J-convex. Take $\Gamma_1(\psi)$ into account which is mentioned in Remark 1.2. Then the exponential J-convexity of $r \mapsto \Gamma_1(h_r)$ holds on I. The function $r \mapsto \Gamma_1(h_r)$ is exponentially convex on I if it is continuous also.*

Corollary 2.6. *Consider a family of continuous functions on $S \subseteq \mathbb{R}$ defined by $\Xi = \{h_r | r \in I \subset \mathbb{R}\}$, such that for every three different points $s_1, s_2, s_3 \in S$ the function $r \mapsto [s_1, s_2, s_3;\ h_r]$ on the interval I is 2-exponentially J-convex. Take $\Gamma_1(\psi)$ into account which is mentioned in Remark 1.2. Then below statements are true:*

i) *Assuming the continuity of the function $r \mapsto \Gamma_1(h_r)$ implies the 2-exponential convexity of $r \mapsto \Gamma_1(h_r)$ on I, which concludes the log-convexity stated as:*

$$\Gamma_1^{t-k}(h_s) \leq \Gamma_1^{t-s}(h_k)\, \Gamma_1^{s-k}(h_t)$$

for $k, s, t \in I$ with $k < s < t$.

ii) *Assume that the function $r \mapsto \Gamma_1(h_r)$ on I is strictly positive and its first order derivative also exists, then for $j \leq s$ and $k \leq t$, $(j, k, s, t \in I)$ yields*

$$\kappa\,(j, k\,; \Gamma_1) \leq \kappa\,(s, t\,; \Gamma_1),$$

where

(7)
$$\kappa\,(j, k\,; \Gamma_1) = \begin{cases} \left(\dfrac{\Gamma_1(h_j)}{\Gamma_1(h_k)}\right)^{\frac{1}{j-k}}, & j \neq k; \\ \exp\left(\dfrac{\frac{d}{dj}\left(\Gamma_1(h_j)\right)}{\Gamma_1(h_j)}\right), & otherwise. \end{cases}$$

Proof. **i)** It directly follows from Remark 2.1 and Theorem 2.4.

ii) (i) follows the log-convexity of function $r \mapsto \Gamma_1(h_r)$ on I, which yields $r \mapsto \log\Gamma_1(h_r)$ is a convex function on I. Now for $j \leq k$ and $s \leq t$, applying [6, Proposition 3.2] we obtained

(8) $$\frac{\log\Gamma_1(h_j) - \log\Gamma_1(h_k)}{j-k} \leq \frac{\log\Gamma_1(h_s) - \log\Gamma_1(h_t)}{s-t}.$$

It yields

$$\kappa\,(j,k\,;\Gamma_1) \leq \kappa\,(s,t\,;\Gamma_1).$$

By applying limit on (8) follows the remaining cases. □

To examine exponential convexity various families of functions are used in the following result. The below lemma is helpful for the construction of exponentially convex functions. Here the proofs are omitted because this result is an easy outcome of remarks and basic examples from [7].

Lemma 2.7. (i) *Consider a self map η_r on $S = \mathbb{R}$, for real positive values r, defined as*

$$\eta_r(s) = \frac{\exp(rs)}{r^2}.$$

This implies the exponential convexity on $(0,\infty)$ of the function $r \mapsto \frac{d^2}{ds^2}\eta_r(s)$ for each s in S.

(ii) *Suppose S is the set of positive real numbers and define a mapping $\zeta_r\colon J \to \mathbb{R}$ $(r > 1)$ by*

$$\zeta_r(s) = \frac{s^r}{r(r-1)}.$$

This implies the exponential convexity on $(1,\infty)$ of the function $r \mapsto \frac{d^2}{ds^2}\zeta_r(s)$ for each s in S.

(iii) *Consider a self map ξ_r on $S = \mathbb{R}^+$, for real positive values $r > 1$, defined as*

$$\xi_r(s) = \frac{r^{-s}}{(\log r)^2}.$$

This implies the exponential convexity on $(1,\infty)$ of the function $r \mapsto \frac{d^2}{ds^2}\xi_r(s)$ for each s in S.

(iv) *Consider a self map ρ_r on $S = \mathbb{R}^+$, for real positive values r, defined as*

$$\rho_r(s) = \exp\frac{-s\sqrt{r}}{r}.$$

This implies the exponential convexity on $(0,\infty)$ of the function $r \mapsto \frac{d^2}{ds^2}\rho_r(s)$ for each s in S.

Remark 2.8. *In defining fundamental inequality of logarithmic convexity these described positive functionals are very useful.*

$$\Gamma_1(\eta_r) = \frac{1}{r^2}\left[a\exp\left(\frac{r(\alpha+\gamma)}{2}\right) + b\exp\left(\frac{r(\beta+\gamma)}{2}\right) - \exp\left(\frac{r(\alpha+\beta)}{2}\right)\right].$$

$$\Gamma_1(\zeta_r) = \frac{1}{r(r-1)2^r}\left[a(\alpha+\gamma)^r + b(\gamma+\beta)^r - (\alpha+\beta)^r\right].$$

$$\Gamma_1(\xi_r) = \frac{1}{2\log r}\left[a\, r^{-\frac{1}{2}(\alpha+\gamma)} + b\, r^{-\frac{1}{2}(\beta+\gamma)} - r^{-\frac{1}{2}(\alpha+\beta)}\right].$$

$$\Gamma_1(\rho_r) = \frac{1}{r}\left[a \exp\left(-\frac{1}{2}(\alpha+\gamma)\sqrt{r}\right) + b\, \exp\left(-\frac{1}{2}(\beta+\gamma)\sqrt{r}\right) - \exp\left(-\frac{1}{2}(\alpha+\beta)\sqrt{r}\right)\right].$$

Theorem 2.9. *Consider the linear functional* $\Gamma_1(\psi)$ *stated in* (3). *Now for* $j = 1, 4$ *and* $j = 2, 3$ *let's define* $\theta_j\colon (0,\infty) \to \mathbb{R}$ *and* $\theta_j : (1,\infty) \to \mathbb{R}$, *respectively, as*

$$\theta_1(r) = \Gamma_1(\eta_r), \qquad \theta_2(r) = \Gamma_1(\zeta_r), \qquad \theta_3(r) = \Gamma_1(\xi_r), \qquad \theta_4(r) = \Gamma_1(\rho_r).$$

We then have:

(i) *For* $j = 1, 4$ *and* $j = 2, 3$, θ_j *functions preserve continuity on* $(0,\infty)$ *and* $(1,\infty)$, *respectively.*

(ii) *For natural number* m, *suppose* $r_j \in (0,\infty)$ $(1 \le j \le m)$ *and* $r_j \in (1,\infty)$ $(1 \le j \le m)$ *for* $j = 1, 4$ *and* $j = 2, 3$, *respectively. This implies that the below matrices are positive semidefinite.*

$$\left[\theta_j\left(\frac{r_i + r_k}{2}\right)\right]_{k,i=1}^{m}.$$

(iii) *The exponential convexity holds for* θ_j *functions on* $(0,\infty)$ *and* $(1,\infty)$ *for* $j = 1,4$ *and* $j = 2,3$, *respectively.*

(iv) *Suppose* $s,t,u \in (0,\infty)$ and $s,t,u \in (1,\infty)$ *for* $j = 1, 4$ *and* $j = 2, 3$, *respectively. It yields*

$$\left(\theta_j(t)\right)^{u-s} \le \left(\theta_j(s)\right)^{u-t}\left(\theta_j(u)\right)^{t-s}.$$

(v) *Assume that* θ_j *functions are strictly positive and their first order derivative also exist on* $(0,\infty)$ *and* $(1,\infty)$ *for* $j = 1, 4$ *and* $j = 2,3$, *respectively. Then for* $i \le s$ *and* $k \le t$, *where* $i,k,s,t \in (0,\infty)$ *and* $i,k,s,t \in (1,\infty)$ *for* $j = 1,4$ *and* $j = 2, 3$, *respectively yield*

$$\kappa\left(i,k\,;\theta_j\right) \le \kappa\left(s,t\,;\theta_j\right),$$

with

(9)
$$\kappa\,(i,k\,;\theta_j) = \begin{cases} \left(\dfrac{\theta_j(i)}{\theta_j(k)}\right)^{\frac{1}{i-k}}, & i \ne k; \\ \exp\left(\dfrac{\frac{d}{di}\left(\theta_j(i)\right)}{\theta_j(i)}\right), & \textit{otherwise}. \end{cases}$$

Proof. (i) The functions $r \mapsto \theta_j(r)$ $(1 \le j \le 4)$ are obviously continuous.

(ii) For natural number m and c_k, c_i are real numbers for $k, i = 1, \cdots, m$. Consider the function Υ_1 on the set $S = \mathbb{R}$ defined as

$$\Upsilon_1(s) = \sum_{k,i=1}^{m} c_k c_i \eta_{\frac{r_k+r_i}{2}}(s).$$

Now for $s \in S$ Lemma 2.7 yields

$$\Upsilon_1''(s) = \sum_{k,i=1}^{m} c_k c_i \frac{d^2}{ds^2} \eta_{\frac{r_k+r_i}{2}}(s) \geq 0.$$

This yields the convexity of Υ_1. Theorem 1.1 results that $\Gamma_1(\Upsilon_1)$ is non negative. It implies that the following matrix is a positive semidefinite matrix

$$\left[\theta_1\left(\frac{r_i + r_k}{2}\right)\right]_{k,i=1}^{m}.$$

is a positive semidefinite matrix.

Analogously, the auxiliary functions Υ_j $(j \in \{2, 3, 4\})$ may be defined that are helpful in proving rest of the positive semidefinite matrices.

(i), (ii) and Lemma 2.3 simply yield (iii) and (iv). Part (iv) is simply used to prove (v).

3. MEAN VALUE THEOREMS

Below lemma is important in proving our results.

Lemma 3.1. [21] *Consider $S = [\alpha, \beta] \subseteq \mathbb{R}$, $\psi \in C^2(S)$. Suppose $\psi : S \to R$, ψ'' is bounded and let $d = \inf_{s \in S} \psi''(s)$, $D = \sup_{s \in S} \psi''(s)$. It implies the convexity of the real functions ψ_1, ψ_2 defined over the set S* as

(10)
$$\psi_1(s) = \frac{D}{2}s^2 - \psi(s)$$

$$\psi_2(s) = \psi(s) - \frac{d}{2}s^2.$$

Theorem 3.2. *Suppose a compact set $S = [\alpha, \beta] (\subseteq \mathbb{R})$ and assume a real function ψ on S, where $\psi \in C^2(S)$. Consider the points $\gamma, \delta \in S$, and a, b, c, d are defined in* (1). *It implies the existence of point $\tau \in S$ such that*

(11)
$$a\psi\left(\frac{\alpha + \gamma}{2}\right) + b\psi\left(\frac{\beta + \gamma}{2}\right) - \psi\left(\frac{\alpha + \beta}{2}\right) = \varepsilon\psi''(\tau),$$

where

$$\varepsilon = \frac{1}{2}\left[a\left(\frac{\alpha+\gamma}{2}\right)^2 + b\left(\frac{\beta+\gamma}{2}\right)^2 - \left(\frac{\alpha+\beta}{2}\right)^2\right]$$

Proof. Suppose $d = \min_{s \in S} \psi''(s)$, $D = \max_{s \in S} \psi''(s)$. Lemma 3.1 follows the convexity of functions $\psi_1, \psi_2 : S \to \mathbb{R}$; the continuity property also holds for ψ_1 and ψ_2. Now using the leftmost inequality of (2) yields

$$a\psi\left(\frac{\alpha + \gamma}{2}\right) + b\psi\left(\frac{\beta + \gamma}{2}\right) - \psi\left(\frac{\alpha + \beta}{2}\right) \leq \varepsilon D,$$

and

$$a\psi\left(\frac{\alpha+\gamma}{2}\right)+b\psi\left(\frac{\beta+\gamma}{2}\right)-\psi\left(\frac{\alpha+\beta}{2}\right)\geq \varepsilon d.$$

Joining the above two inequalities and using the fact that second order derivative of ψ is continuous, results the existence of a point τ in S with $d \leq \psi''(\tau) \leq D$. This proves the required result.

Theorem 3.3. *Suppose a compact set* $S = [\alpha, \beta](\subseteq \mathbb{R})$ *and assume two real functions* χ, ψ *on* S*, where* $\chi, \psi \in C^2(S)$*. Consider the points* $\gamma, \delta \in S$*, and* a, b, c, d *are defined in* (1)*. It implies the existence of point* $\tau \in S$ *such that*

(12)
$$\psi''(\tau)\left[a\chi\left(\frac{\alpha+\gamma}{2}\right)+b\chi\left(\frac{\beta+\gamma}{2}\right)-\chi\left(\frac{\alpha+\beta}{2}\right)\right]$$
$$=\chi''(\tau)\left[a\psi\left(\frac{\alpha+\gamma}{2}\right)+b\psi\left(\frac{\beta+\gamma}{2}\right)-\psi\left(\frac{\alpha+\beta}{2}\right)\right].$$

Proof. Define a function $\vartheta \in C^2(S)$ by $\vartheta = e_1\chi - e_2\psi$, where

(13)
$$e_1 = a\psi\left(\frac{\alpha+\gamma}{2}\right)+b\psi\left(\frac{\beta+\gamma}{2}\right)-\psi\left(\frac{\alpha+\beta}{2}\right)$$

and

(14)
$$e_2 = a\chi\left(\frac{\alpha+\gamma}{2}\right)+b\chi\left(\frac{\beta+\gamma}{2}\right)-\chi\left(\frac{\alpha+\beta}{2}\right).$$

As the function $\vartheta \in C^2(S)$ and implementing this function to Theorem 3.2 implies the existence of a point $\tau \in S$ such as

(15)
$$a\vartheta\left(\frac{\alpha+\gamma}{2}\right)+b\vartheta\left(\frac{\beta+\gamma}{2}\right)-\vartheta\left(\frac{\alpha+\beta}{2}\right)=\varepsilon\vartheta''(\tau).$$

The expression on right side of this equation is non zero, whereas the one on the left side is zero. Thus it follows, $\vartheta''(\tau) = 0$ concluding the required result.

Remark 3.4. *We may describe different types of means by applying* (12) *under the assumption that* χ''/ψ'' *is invertible.*

Such as,

(16)
$$\tau = \left(\frac{\chi''}{\psi''}\right)^{-1}\left(\frac{\Gamma_1(\chi)}{\Gamma_1(\psi)}\right).$$

Applying mean value Theorem 3.3 (Cauchy kind) on $\chi = \eta_j$, $\psi = \eta_k$ (given by Lemma 2.7). This implies

$$Q(j,k;\,\Gamma_1) = \log\kappa(j,k;\Gamma_1)$$

provide

$$\alpha \leq Q(j,k;\Gamma_1) \leq \beta.$$

Thus $Q(j,k;\,\Gamma_1)$ is a mean. Now suppose j, k, s and t are real numbers such as $j \leq k, s \leq t$ then Theorem 2.9 results that this mean is monotonic.

$$\kappa\,(j,k\,;\Gamma_1)=\begin{cases}\left(\dfrac{\Gamma_1(\eta_j)}{\Gamma_1(\eta_k)}\right)^{\frac{1}{j-k}}, & j\neq k;\\ \exp\left(\dfrac{\Gamma_1(id\cdot\eta_j)}{\Gamma_1(\eta_j)}\right).\exp\left(-\dfrac{2}{j}\right), & j=k\neq 0.\end{cases}$$

Furthermore, applying mean value Theorem 3.3 (Cauchy kind) on $\chi=\zeta_j\,,\psi=\zeta_k$ (given by Lemma 2.7). This implies the existence of an element $\tau\in S$ so that

$$\tau^{j-k}=\frac{\Gamma_1(\zeta_j)}{\Gamma_1(\zeta_k)}.$$

For distinct points j,k, we obtain

$$\alpha\leq\left(\frac{\Gamma_1(\zeta_j)}{\Gamma_1(\zeta_k)}\right)^{\frac{1}{j-k}}\leq\beta$$

provided that $\tau\mapsto\tau^{j-k}$ is invertible. This provides $\kappa(j,k;\Gamma_1)$ is a mean which is monotonic as well, where

$$\kappa\,(j,k\,;\Gamma_1)=\begin{cases}\left(\dfrac{\Gamma_1(\eta_j)}{\Gamma_1(\eta_k)}\right)^{\frac{1}{j-k}}, & j\neq k;\\ \exp\left(-\dfrac{\Gamma_1(\zeta_0\cdot\zeta_j)}{\Gamma_1(\zeta_j)}\right).\exp\left(\dfrac{1-2j}{j(j-1)}\right), & j=k\neq 1.\end{cases}$$

Remark 3.5. *Analogous result can also be constructed for* $\Gamma_j(\psi),\ j\ =\ 2,3,4$ *stated in Remark 1.2.*

4. ACKNOWLEDGEMENTS

This work was supported by National University of Sciences and Technology (NUST), Islamabad and Higher Education Commission of Pakistan.

5. REFERENCES

1. Pavić, Z. Improvements of the Hermite-Hadamard inequality. *Journal of Inequalities Applications* 2015(1): 1-11 (2015).
2. Dragomir, S.S., J.E. Pečarić, & L.E. Persson. Properties of some functionals related to Jensen's inequality. *Acta Mathematica Hungarica* 70(1-2): 129-143 (1996).
3. Mercer, A. McD. An "error term" for the Ky Fan inequality. *Journal of Mathematical Analysis and Applications* 220(2): 774-777 (1998).
4. Mercer, A. McD. Some new inequalities involving elementary mean values. *Journal of Mathematical Analysis and Applications* 229: 677-681 (1999).
5. Pečarić, J., I. Perić and H. M. Srivastava, A family of Cauchy type mean-value theorems. *Journal of Mathematical Analysis and Applications* 306: 730-739 (2005).
6. Pečarić, J., J. Perić, Improvement of the Giaccardi and the Petrovi´c inequality and related Stolarsky type means. *Annals of the University of Craiova-Mathematics and Computer Science Series* 39(1): 65-75 (2012).
7. Jekšetić, J., & J. Pečarić. Exponential Convexity method, *Journal of Convex Analysis* 20(1): 181-187 (2013).
8. Anwar, M., N. Latif, & J. Pečarić. Positive semidefinite matrices, exponential convexity for majorization, and related Cauchy means. *Journal of Inequalities and Applications* 2010: 1-8 (2010).
9. Bernstein, S.N. Sur les fonctions absolument monotones. *Acta Mathematica* 52(1): 1-66 (1929).
10. Butt, S.I., R. Jakšić, L. Kvesić, & J. Pečarić. *n*-Exponential convexity of weighted Hermite-Hadamards inequality. *Journal of Mathematical Inequalities* 8(2): 299-311 (2014).

11. Butt, S.I., J. Pečarić, I. Perić, & M. Praljak. Multidimensional reversed Hardy type inequalities for monotone functions. *Asian-European Journal of Mathematics* 7(4): 1450055 (2014).
12. Iqbal, S., K.K. Himmelreich, J. Pečarić, & D. Pokaz. *n*-Exponential convexity of Hardy-type and Boas-type functionals. *Journal of Mathematical Inequalities* 7(4): 739-750 (2013).
13. Jekšetić, J., R. Naeem, & J. Pečarić. Exponential convexity for Jensen's inequality for norms. *Journal of Inequalities and Applications* 2016(1): 1-8 (2016).
14. Khalid, S., & J. Pečarić. On the refinements of the Hermite-Hadamard inequality. *Journal of Inequalities and Applications* 2012(1): 1-14 (2012).
15. Khalid, S., & J. Pečarić. On the refinements of the integral Jensen-Steffensen inequality, *Journal of Inequalities and Applications* 2013(1): 1-18 (2013).
16. Khan, A.R., J. Pečarić, & M.R. Lipanović. *n*-Exponential convexity for Jensen-type inequalities. *Journal of Mathematical Inequalities* 7(3): 313-335 (2013).
17. Khan, K.A., A. Nosheen, & J. Pečarić. *n*-Exponential convexity of some dynamic Hardy-type functionals. *Journal of Mathematical Inequalities* 8(2): 331-347 (2014).
18. Pečarić, J., A. Perušić, & A. Vukelić. Generalisations of Steffensen's inequality via Fink identity and related results. *Advances in Inequalities and Applications* 2014: Article ID 9 (2014).
19. Pečarić, J., A. Perušić, K. Smoljak, Mercer & Wu Srivastava. Generalisations of Steffensen's inequality. *Applied Mathematics and Computation* 219(21): 10548-10558 (2013).
20. Widder, D.V. Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral. *Bulletin of American Mathematical Society* 40(4): 321-326 (1934).
21. Pečarić, J., F. Proschan and Y. L. Tong. *Convex Functions, Partial Orderings, and Statistical Applications*. *Mathematics in Science and Engineering Volume 187*. Academic Press, Boston, MA (1992).