Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 54 (2): 207–218 (2017)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

On the Existence of Periodic Solutions to Certain Non-linear Differential Equations of Third Order

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Abstract: We investigated the existence of periodic solutions to a kind of non-linear differential equations of third order. We prove two new theorems on the existence of periodic solutions to the equation considered. The technique of proof involves the Leray-Schauder degree theory. The results obtained include and improve some results found in the literature.

Keywords: Nonlinear differential equation, third order, existence of periodic solutions,

Leray-Schauder degree theory

2010 AMS (MOS) Subject Classifications: 34C25, 47N20

1. INTRODUCTION

In this paper, we consider third-order nonlinear differential equations of the form

$$\dddot{x} + f(\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1}$$

where $f:\Re \to \Re$, $g:\Re \to \Re$, $h:\Re \to \Re$ and $f:\Re^{+} \times \Re \times \Re \times \Re \to \Re$ are continuous functions, $\Re = (-\infty, \infty)$, $\Re^{+} = [0, \infty)$, and p is a T- periodic function, that is,

$$p(t+T, x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}),\ T > 0,\ T \in \Re.$$

We investigate here the existence of periodic solutions of Eq. (1).

In 1974, Reisisg et al. [6] considered the following non-linear differential equations of third order

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t), \tag{2}$$

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + cx = p(t) \tag{3}$$

and

$$\dddot{x} + f(\ddot{x}) + b\dot{x} + cx = p(t), \tag{4}$$

where a, b and c are real constants.

Reissig et al. [6] discussed the existence of periodic solutions to Eq. (2) – Eq. (4) by using the Leray-Schauder degree theory.

Then, in 1991, Andres and Vlček [2] investigated the existence of periodic solutions of the following triad differential equations of third order

Received, August 2016; Accepted, May 2017
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$$\dddot{x} + L(t,x) = a(\ddot{x}),$$

$$\dddot{x} + L(t,x) = b(\dot{x})$$

and

$$\dddot{x} + L(t,x) = c(\ddot{x}),$$

where

$$L(t,x) = f(t)\ddot{x} + g(t)\dot{x} + h(x) + p(t) + q(t,x,\dot{x},\ddot{x}).$$

They benefited from the standard Leray-Schauder alternative method to perform the proofs of the results of [2].

Then, in 1996, Andres [3]) investigated the existence of periodic solutions to Eq. (2)–Eq. (4).The results of [3] have been proved by means of the Lyapunov's second method.

Later, in 1999, Huang [5] generalized the results of [1–3] and [6] to nonlinear differential equation of third order

$$\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t) \tag{5}$$

by help of Leray-Schauder degree theory.

Further, Tunç [8] studied the existence of periodic solutions to the following nonlinear differential equation of third order

$$\dddot{x} + c_2(t)\ddot{x} + c_1(t)\dot{x} + f(t,x) = p(t,x,\dot{x},\ddot{x})$$

by using the Leray-Schauder degree theory.

For some the other works related to the existence of the periodic solutions of certain non-linear differential equations of third order, we refer the readers to various papers of [4, 7, 9, 10], [11].

It is worth mentioning to state some results of [1–3, 6].

Theorem A. Eq. (2), Eq. (3) or Eq. (4) has at least one *T*-periodic solution if they satisfy the following assumptions, $(A1)$, $(A2)$ and $(A3)$, respectively:

$(A1)$ $b < \dfrac{4\pi^2}{T^2}$,

and there is a constant $R > 0$ such that

$$[h(x) - \overline{p}]x < 0$$

or

$$[h(x) - \overline{p}]x > 0,$$

for $|x| > R$, where $\overline{p} = \dfrac{1}{T}\int_0^T p(t)dt$,

$(A2)$ $ac < 0$,

$(A3)$ $b < \dfrac{4\pi^2}{T^2}$, $c \neq 0$.

We now consider the following linear and non-homogeneous differential equation of third order with constant coefficients a, b and c:

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t). \tag{6}$$

It is well known from the relevant literature that the sufficient condition to guarantee the existence of *T*-periodic solutions for Eq. (6) is that the corresponding linear homogeneous equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \tag{7}$$

does not have non-trivial *T*-periodic solution.

The characteristic equation of Eq. (7) is

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0. \tag{8}$$

The basic assumption for Eq. (6) does not have a non-trivial T -periodic solution is

$$\lambda \neq \frac{2k\pi i}{T},$$

where k is an integer number.

From $\lambda = \dfrac{2k\pi i}{T}$ and Eq. (8), we know that

$$\frac{c}{a} = \frac{4k^2\pi^2}{T^2}$$

and

$$b = \frac{4k^2\pi^2}{T^2}.$$

Hence, we can say that if one of the following conditions

$(H1)$ $\dfrac{c}{a} \neq \dfrac{4k^2\pi^2}{T^2}$

or

$(H2)$ $b \neq \dfrac{4k^2\pi^2}{T^2}$

holds, then Eq. (7) does not have a non-trivial *T*-periodic solution. Therefore, these conditions guarantee the existence of *T*-periodic solutions of Eq. (6).

In order to perform the main problems of this paper, we apply the following standard the Leray-Schauder 's theorem (Reisisg et al. [6], Theorem 1.38).

Theorem B. Let

$$L(t,x) = f(t)\ddot{x} + g(t)\dot{x} + h(x) + p(t) + q(t,x,\dot{x},\ddot{x}).$$

If all $\omega-$ periodic solutions of the one parametric family of equations

$$\dddot{x} + \mu L(t,x) + (1-\mu)ex = \mu w(x,\dot{x},\ddot{x}),\ \mu \in (0,1),$$

where $w(x,\dot{x},\ddot{x})$ denotes $a(\ddot{x})$ or $b(\dot{x})$ or $c(x)$ and e is a suitable non-zero real, are uniformly priori bounded together with their first, second derivatives, independently of $\mu \in (0,1),$ and the linear equation, resulting from the former differential equation for $\mu = 0$, has no non-trivial $\omega-$periodic solution, then the equation obtained from the former equation for $\mu = 1$ admits a harmonic.

Proof. See [6, Theorem 1.38].

The aim of this paper is to study the existence of *T*-periodic solutions of Eq. (1). We benefit from the Leray-Schauder degree theory to obtain sufficient conditions which guarantee the existence of *T*-periodic solutions for Eq. (1).

It is worth mentioning that when we compare Eq. (6) with Eq. (1), it can be seen that Eq. (1) is a non-linear generalization of Eq. (6). In fact, it is natural to expect that the assumptions to be established here for the existence of *T*-periodic solutions of Eq. (1) can be a non-linear generalization of assumptions $(H1)$ or $(H2)$.

In addition, it follows that Eq. (1) includes and improves Eq.(2)-Eq. (3) and Eq. (4)-Eq. (5) when $f(\ddot{x}) = \ddot{x}$.

Further, Eq. (1) is a different model from those discussed in the literature. This is the novelty and newness of the problem considered here.

Eq. (1) can be rewritten as

$$\dddot{x} + f(\dot{x})\ddot{x} + \frac{g(\dot{x})}{\dot{x}}\dot{x} + \frac{h(x)}{x}x = p(t, x, \dot{x}, \ddot{x}), \tag{9}$$

where $x \neq 0$ and $\dot{x} \neq 0$.

Then, the following question might be expected. Namely, if we take, respectively, $f(\dot{x})$, $\frac{g(\dot{x})}{\dot{x}}$, $\frac{h(x)}{x}$ as a, b, c of Eq. (6), and satisfy the similar conditions to $(H1)$, (H_2), can we conclude that Eq.(1) has the *T*-periodic solutions? The answer is no. However, when Eq. (1) satisfies similar conditions to $(H1)$ or $(H2)$ and additional limited conditions, it can be proved the existence of *T*-periodic solutions of Eq. (1).

The motivation of this paper comes from the results established by Huang [5], the related ones in the mentioned books, the papers mentioned and that found in theirs references. The main purpose of this paper is to investigate the existence of periodic solutions of Eq. (1) by using the Leray-Schauder degree theory. By this paper, we extend and improve the results of Huang [5] for Eq. (1).This paper has also a contribution to the subject, and it may be useful for researchers working on the qualitative properties of solutions of differential equations of third order. In view of all the mentioned information, it can be checked the novelty and originality of the present paper.

2. EXISTENCE OF PERIODIC SOLUTIONS

The following lemma is needed in the proof of the main result.

Lemma 1. Let $\xi(t) \in C^1$, $\dot{\xi}(t) \in L^2[0,T]$ and M_1 and M_2 be some positive constants. If

$$\int_0^T |\xi(t)|^2 \, dt \leq M_1 \, , \int_0^T |\dot{\xi}(t)|^2 \, dt \leq M_2, \tag{10}$$

then there exists a constant $M > 0$ (only relative to M_1, M_2) such that $|\xi(t)| \leq M$ for $t \in [0,T]$.

Proof. Let $L = \max\{M_1, M_2\}$ and $M^2 = (2 + T^{-1})L$. Hence, we have $|\xi(t)| \le M$. If the former estimate is not true, then there may be $t_0 \in [0,T]$ such that $\xi^2(t_0) > M^2$. In view of the estimates given by (10), it follows that there is $t_1 \in [0,T]$ such that $\xi^2(t_1) \le \frac{L}{T}$. Hence, it is notable that

$$2L < [\xi^2(t_0) - \xi^2(t_1)] = 2\int_{t_1}^{t_0} \xi(t)\xi'(t)dt$$

$$\le 2\int_0^T |\xi(t)|\ |\xi'(t)|dt$$

$$\le 2\left(\int_0^T |\xi(t)|^2\ dt\right)^{\frac{1}{2}}\left(\int_0^T |\xi'(t)|^2\ dt\right)^{\frac{1}{2}} \le 2L,$$

which is a contradiction. Thus, this completes the proof of Lemma 1.

Lemma 2. Let $\xi(t) \in C^2$ be the periodic function of period T. Then

$$\int_0^T (\dot{\xi}(t))^2\, dt \le \frac{T^2}{4\pi^2}\int_0^T (\ddot{\xi}(t))^2\, dt. \tag{11}$$

Proof. Inequality (11) is a result of Wirtinger's inequality. Therefore, we leave its proof.

The following theorem is our first main result on the existence of *T*-periodic solutions of Eq. (1).

Theorem 1. We assume that there exist constants $a,\ c,\ (a \ne 0,\ c \ne 0),\ m\ (> 0)$, and $M > 0$ such that the following conditions hold:

$(A1)\ |cx - h(x)| \le M$,

$(A2)\ a(a - f(y)) \le 0$,

$(A3)\ \frac{c}{a} < \frac{4\pi^2}{T^2}$,

$(A4)\ \ |p(t, \xi, \dot{\xi}, \ddot{\xi})| \le |\theta(t)| \le m$,

where $\theta(t)$ is a continuous function for all $t \in [0,T]$. Then, Eq.(1) has at least one *T*-periodic solution.

Proof. Eq. (1) can be rewritten as

$$\dddot{x} + a\ddot{x} + cx = p(t, x, \dot{x}, \ddot{x}) + f_1(\dot{x})\ddot{x} + g_1(\dot{x}) + h_1(x), \tag{12}$$

where

$$f_1(\dot{x}) = a - f(\dot{x}),$$

$$g_1(\dot{x}) = -g(\dot{x}),$$

$$h_1(x) = cx - h(x).$$

Consider the following differential equation

$$\dddot{x}+a\ddot{x}+cx=\lambda[p(t,x,\dot{x},\ddot{x})+f_1(\dot{x})\ddot{x}+g_1(\dot{x})+h_1(x)], \tag{13}$$

where $\lambda\in(0,1)$.

For $\lambda=1$, Eq. (13) becomes equal to Eq. (1) and when $\lambda=0$, becomes equal to differential equation

$$\dddot{x}+a\ddot{x}+cx=0,$$

which has only one trivial *T*-periodic solution.

Regarding to the Leray-Schauder degree theory (see Theorem B, that is, [6, Theorem 1.38]), it is only needed to prove that all the *T*-periodic solutions $\xi(t)$ and $\dot{\xi}(t),\ddot{\xi}(t)$ of Eq. (13) are uniformly bounded for all $\lambda\in[0,1]$.

Let $\xi(t)$ be the *T*-periodic solution of Eq. (13). Multiplying Eq. (13) with $\dfrac{\ddot{\xi}(t)}{a}$ and then integrating from 0 to *T*, we have

$$\frac{1}{a}\int_0^T\ddot{\xi}(t)\dddot{\xi}(t)dt+\int_0^T(\ddot{\xi}(t))^2dt+\frac{c}{a}\int_0^T\xi(t)\ddot{\xi}(t)dt$$

$$=\frac{\lambda}{a}\int_0^T\ddot{\xi}(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))+f_1(\dot{\xi}(t))\ddot{\xi}(t)+g_1(\dot{\xi}(t))+h_1(\xi(t))]dt.$$

Applying integration by parts, we obtain

$$\int_0^T(\ddot{\xi}(t))^2dt-\frac{c}{a}\int_0^T(\dot{\xi}(t))^2dt=\frac{\lambda}{a}\Big[\int_0^T\ddot{\xi}(t)p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))dt$$

$$+\int_0^T(\ddot{\xi}(t))^2f_1(\dot{\xi}(t))dt+\int_0^T\ddot{\xi}(t)h_1(\xi(t))dt\Big]. \tag{14}$$

It is well known from Lemma 2 that if $\xi(t)\in C^2$ is a periodic function of period T, then

$$\int_0^T(\dot{\xi}(t))^2dt\le\frac{T^2}{4\pi^2}\int_0^T(\ddot{\xi}(t))^2dt.$$

When we use this inequality in (14), we obtain

$$\int_0^T(\ddot{\xi}(t))^2dt=\frac{c}{a}\int_0^T(\dot{\xi}(t))^2dt+\frac{\lambda}{a}\int_0^T\ddot{\xi}(t)p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))dt$$

$$+\frac{1}{a}\int_0^T(\ddot{\xi}(t))^2f_1(\dot{\xi}(t))dt+\frac{1}{a}\int_0^T\ddot{\xi}(t)h_1(\xi(t))dt$$

$$\le\frac{cT^2}{4a\pi^2}\int_0^T(\ddot{\xi}(t))^2dt+\frac{1}{a}\int_0^T\ddot{\xi}(t)p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))dt$$

By re-arranging the former inequality, we have

$$\int_0^T(\ddot{\xi}(t))^2dt-\frac{cT^2}{4a\pi^2}\int_0^T(\ddot{\xi}(t))^2dt$$

$$\leq \frac{1}{a}\int_0^T \dddot{\xi}(t)p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))dt + \frac{1}{a^2}\int_0^T a\, f_1(\dot{\xi}(t))(\dddot{\xi}(t))^2 dt$$

$$+\frac{1}{a}\int_0^T \dddot{\xi}(t)h_1(\xi(t))dt$$

Then, it follows from condition $(A2)$ of Theorem 1 that

$$\left(1-\frac{cT^2}{4a\pi^2}\right)\int_0^T (\dddot{\xi}(t))^2 dt \leq \frac{1}{a}\int_0^T \dddot{\xi}(t)p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))dt + \frac{1}{a}\int_0^T \dddot{\xi}(t)h_1(\xi(t))dt$$

so that

$$\left(1-\frac{cT^2}{4a\pi^2}\right)\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \leq \frac{1}{|a|}\left[\int_0^T \left|\dddot{\xi}(t)\right|\left|p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))\right|dt + \int_0^T \left|\dddot{\xi}(t)\right|\left|h_1(\xi(t))\right|dt\right].$$

By Lemma 2 and the conditions of Theorem 1, we have

$$\left(1-\frac{cT^2}{4a\pi^2}\right)\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \leq \frac{m+M}{|a|}\left(\int_0^T \left|\dddot{\xi}(t)\right|dt\right) \leq \frac{m+M}{|a|}\sqrt{T}\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

That is,

$$\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}} \leq \left(1-\frac{cT^2}{4a\pi^2}\right)^{-1}\frac{m+M}{|a|}\sqrt{T}.$$

Let

$$\left(\left(1-\frac{cT^2}{4a\pi^2}\right)^{-1}\frac{m+M}{|a|}\sqrt{T}\right)^2 = M_1.$$

Then

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \leq M_1,$$

where $M_1 > 0$ is independent of λ.

Also, in view of the last inequality, it follows from Lemma 2 that there exists a constant $M_2 > 0$ such that

$$\int_0^T \left|\dot{\xi}(t)\right|^2 dt \leq M_2 .$$

From Lemma 1, it follows that there exists $M_3 > 0$ such that

$$\left|\dot{\xi}(t)\right| \leq M_3 .$$

Multiplying Eq. (13) with $\ddddot{\xi}(t)$ and then integrating from 0 to *T*, we have

$$\int_0^T (\ddddot{\xi}(t))^2 dt + a\int_0^T \ddddot{\xi}(t)\dddot{\xi}(t)dt + c\int_0^T \xi(t)\ddddot{\xi}(t)dt$$

$$= \lambda \int_0^T \dddot{\xi}(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]dt.$$

Applying integration by parts, we obtain

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt = \lambda \int_0^T \dddot{\xi}(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]dt.$$

Hence, it follows that

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \le \int_0^T \left|\dddot{\xi}(t)\right|\left|p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))\right| dt$$

$$+\int_0^T \left|f_1(\dot{\xi}(t))\right|\left|\ddot{\xi}(t)\right|\left|\dddot{\xi}(t)\right| dt + \int_0^T \left|g_1(\dot{\xi}(t))\right|\left|\dddot{\xi}(t)\right| dt + \int_0^T \left|h_1(\xi(t))\right|\left|\dddot{\xi}(t)\right| dt.$$

For $\left|\dot{\xi}(t)\right| \le M_3$, it is clear that there exist constants $M_4, M_5 > 0$ such that

$$\left|g_1(\dot{\xi})\right| \le M_4,$$

$$\left|f_1(\dot{\xi})\right| = \left|a - f(\dot{\xi})\right| \le M_5.$$

And also we know from Theorem 1 that

$$\left|h_1(x)\right| = \left|cx - h(x)\right| \le M.$$

Then, we have

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \le (m + M + M_4)\int_0^T \left|\dddot{\xi}(t)\right| dt \le (m + M + M_4)\sqrt{T}\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Therefore, there exists $M_6 > 0$ such that

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \le M_6.$$

By Lemma 1, it follows that there exists a constant $M_7 > 0$ such that $\left|\ddot{\xi}(t)\right| \le M_7$. Multiplying Eq. (13) with $\xi(t)$ and then integrating from 0 to T, we have

$$\int_0^T \xi(t)\dddot{\xi}(t)dt + a\int_0^T \xi(t)\ddot{\xi}(t)dt + c\int_0^T (\xi(t))^2 dt$$

$$= \lambda \int_0^T \xi(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]dt.$$

Applying integrating by parts, we obtain

$$-a\int_0^T (\dot{\xi}(t))^2 dt + c\int_0^T (\xi(t))^2 dt$$

$$= \lambda \int_0^T \xi(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]dt.$$

Hence, it can be followed that

$$\int_0^T \left|\xi(t)\right|^2 dt \le \frac{1}{|c|}\Big[\int_0^T \left|p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t))\right|\left|\xi(t)\right| dt + \int_0^T \left|f_1(\dot{\xi}(t))\right|\left|\ddot{\xi}(t)\right|\left|\xi(t)\right| dt$$

$$+\int_0^T |g_1(\dot{\xi}(t))||\xi(t)|\,dt + \int_0^T |h_1(\xi(t))||\xi(t)|\,dt \Big] + \frac{|a|}{|c|}\int_0^T |\dot{\xi}(t)|\,dt.$$

By noting the assumptions of Theorem 1, we get

$$\int_0^T |\xi(t)|^2\,dt \le \frac{m+M+M_4}{|c|}\int_0^T |\xi(t)|\,dt + \frac{(M_5+|a|)}{|c|}\int_0^T |\dot{\xi}(t)|^2\,dt$$

$$\le \frac{m+M+M_4}{|c|}\sqrt{T}\left(\int_0^T |\xi(t)|^2\,dt\right)^{\frac{1}{2}} + \frac{(M_5+|a|)}{|c|}\int_0^T |\dot{\xi}(t)|^2\,dt.$$

Then, in view of the assumptions of Lemma 2, it is obvious that

$$\int_0^T |\xi(t)|^2\,dt \le \frac{m+M+M_4}{|c|}\sqrt{T}\left(\frac{T^2}{4\pi^2}\int_0^T |\dot{\xi}(t)|^2\,dt\right)^{\frac{1}{2}} + \frac{(M_5+|a|)}{|c|}\int_0^T |\dot{\xi}(t)|^2\,dt.$$

Since

$$\int_0^T |\dot{\xi}(t)|^2\,dt \le M_2,$$

then right hand side of the former inequality is bounded and there exists $M_8 > 0$ such that

$$\int_0^T |\xi(t)|^2\,dt \le M_8.$$

Furthermore, we conclude from Lemma 1 that there exists a constant $M_9 > 0$ such that

$$|\xi(t)| \le M_9.$$

Since M_i, $(i = 1,2,....,9)$, are independent of λ, we know $|\xi(t)|$, $|\dot{\xi}(t)|$ and $|\ddot{\xi}(t)|$ are uniformly bounded to λ. Then, Eq.(1) has at least one *T*-periodic solution. This completes the proof of Theorem 1.

Our second main result is the following theorem on the existence of *T*-periodic solutions of Eq. (1).

Theorem 2. In addition to assumptions $(A1)$ and $(A4)$ of Theorem 1, (except assumptions $(A2)$ and $(A3)$), we assume there exist constants b, $c\ (c \neq 0)$ and α, β, $M > 0$ such that the following conditions hold:

$(B1)\ |by - g(y)| \le \alpha|y| + \beta,$

$(B2)\ b < \dfrac{4\pi^2}{T^2}.$

Then, there exists a constant $\alpha_0 > 0$, when $\alpha < \alpha_0$, Eq.(1) has at least one *T*-periodic solution.

Proof. It is clear from Eq. (1) that

$$\dddot{x} + b\dot{x} + cx = p(t,x,\dot{x},\ddot{x}) + f_1(\dot{x})\ddot{x} + g_1(\dot{x}) + h_1(x),$$

where

$$f_1(\dot{x}) = -f(\dot{x}),$$

$$g_1(\dot{x}) = b\dot{x} - g(\dot{x}),$$

$$h_1(x) = cx - h(x).$$

We now consider the following differential equation

$$\dddot{x} + b\dot{x} + cx = \lambda[p(t,x,\dot{x},\ddot{x}) + f_1(\dot{x})\ddot{x} + g_1(\dot{x}) + h_1(x)], \tag{15}$$

where $\lambda \in [0,1]$.

Similar to Theorem 1, we have only to prove that all the T-periodic solutions $\xi(t)$, $\dot{\xi}(t)$ and $\ddot{\xi}(t)$ of Eq. (15) are uniformly bounded for all $\lambda \in [0,1]$.

Let $\xi(t)$ be T-periodic solution of Eq. (15), that is,

$$\dddot{\xi}(t) + b\dot{\xi}(t) + c\xi(t)$$
$$= \lambda[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]. \tag{16}$$

Multiplying Eq. (16) by $\dddot{\xi}(t)$ and then integrating from 0 to T, we have

$$\int_0^T (\dddot{\xi}(t))^2 dt - b\int_0^T (\ddot{\xi}(t))^2 dt$$
$$= \lambda\int_0^T \dddot{\xi}(t)[p(t,\xi(t),\dot{\xi}(t),\ddot{\xi}(t)) + f_1(\dot{\xi}(t))\ddot{\xi}(t) + g_1(\dot{\xi}(t)) + h_1(\xi(t))]dt.$$

Thus, by the assumptions of Theorem 2 and Lemma 2, we can easily obtain

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \le |b|\frac{T^2}{4\pi^2}\int_0^T \left|\dddot{\xi}(t)\right|^2 dt + (m+M)\int_0^T \left|\dddot{\xi}(t)\right| dt + \alpha\int_0^T \left|\dddot{\xi}(t)\right|\left|\dot{\xi}(t)\right| dt + \beta T$$

$$\le |b|\frac{T^2}{4\pi^2}\int_0^T \left|\dddot{\xi}(t)\right|^2 dt + (m+M)\sqrt{T}\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}}$$

$$+\alpha\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \left|\dot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}} + \beta T$$

$$\le \left(\frac{(|b|+\alpha)T^2}{4\pi^2}\right)\int_0^T \left|\dddot{\xi}(t)\right|^2 dt + (m+M)\sqrt{T}\left(\int_0^T \left|\dddot{\xi}(t)\right|^2 dt\right)^{\frac{1}{2}} + \beta T.$$

Hence, it follows that there exist constants $\alpha_0 > 0,\ M_1 > 0,$ when $\alpha < \alpha_0,$ such that

$$\int_0^T \left|\dddot{\xi}(t)\right|^2 dt \le M_1,$$

where M_1 independent of λ.

The rest of the proof can be completed by following a similar way as shown in the proof of Theorem 1. Therefore, we omit the details.

Example. We consider the following third order nonlinear differential equation

$$\dddot{x} + (2 + e^{-\dot{x}^2})\ddot{x} + \sin\dot{x} + (2x - e^{-x^2}) = \sin 2t, \tag{17}$$

which is a special case of Eq. (1).

When we compare the former equation, Eq. (17), with Eq. (1), it follows that

$$f(\dot{x}) = 2 + e^{-\dot{x}^2},$$
$$g(\dot{x}) = \sin \dot{x},$$
$$h(x) = 2x - e^{-x^2},$$
$$p(t, x, \dot{x}, \ddot{x}) = \sin 2t.$$

Then, Eq. (17) can be rewritten as

$$\dddot{x} + \ddot{x} + 2x = \sin 2t + f_1(\dot{x})\ddot{x} + g_1(\dot{x}) + h_1(x),$$

where

$$a = 1,\ c = 2,$$
$$f_1(\dot{x}) = a - f(\dot{x}) = 1 - (2 + e^{-\dot{x}^2}) = -(1 + e^{-\dot{x}^2}),$$
$$g_1(\dot{x}) = -g(\dot{x}) = -\sin \dot{x},$$
$$h_1(x) = cx - h(x) = 2x - (2x - e^{-x^2}) = e^{-x^2}.$$

Therefore, we can see that

$$\dddot{x} + \ddot{x} + 2x = \sin 2t - (1 + e^{-\dot{x}^2})\ddot{x} - \sin \dot{x} + e^{-x^2}.$$

Hence, it is clear that

$$|cx - h(x)| \le \left|e^{-x^2}\right| \le 1 = M, \quad M > 0,$$
$$a(a - f(y)) = -(1 + e^{-y^2}) \le 0,$$
$$\frac{c}{a} = 2 < \frac{4\pi^2}{T^2},$$

where $T = \pi$.

Thus, all the assumptions of Theorem 1 hold. Thus, we can conclude that Eq. (17) has at least one T-periodic solution.

3. CONCLUSIONS

We consider a specific non-linear differential equation of third order. We describe certain sufficient conditions guaranteeing the existence of at the least one periodic solution for the equation considered. We prove two new theorems on the subject by the help of Leray-Schauder degree theory. The results obtained essentially complement, extend and improve some well-known results in the literature to a more general non-linear case.

4. ACKNOWLEDGEMENTS

The authors would like to express sincere appreciation to the anonymous referees for their valuable comments and suggestions which have led to an improvement in the presentation of the paper.

5. REFERENCES

1. Afuwape, U.A., P. Omari & F. Zanolini. Nonlinear perturbations of differential operators with nontrivial kernel and applications to third-order periodic boundary value problems. *Journal of Mathematical Analysis and Applications* 143(1): 35–56 (1989).

2. Andres, J. & V. Vlček. Periodic solutions of the third order parametric differential equations involving large nonlinearities. *Mathematica Slovaca* 41(4): 337–349 (1991).
3. Andres, J. Existence, uniqueness, and instability of large-period harmonics to the third-order nonlinear ordinary differential equations. *Journal of Mathematical Analysis and Applications* 199 (2): 445–457 (1996).
4. Guo Y.Q. & X.T. Du. Periodic solutions of third-order functional differential equations. *Journal of Mathematical Study* 36 (1): 37–42 (2003).
5. Huang, X. On the existence of periodic solutions for the third-order nonlinear ordinary differential equations. *Applied Mathematics – A Journal of Chinese Universities. Series B* 14(2): 125-130 (1999).
6. Reissig, R., G. Sansone, & R. Conti. *Non-linear Differential Equations of Higher Order*. Noordhoff International Publishing, Leyden (in German) (1974).
7. Tejumola, H.O. On the existence of periodic solutions of some third-order nonlinear differential equations with elay. Special issue in honour of Professor James O. C. Ezeilo. *Journal of Nigerian Mathematical Society* 11(3): 19–28 (1992).
8. Tunç, C. On existence of periodic solution to certain nonlinear third order differential equations. *Proyecciones* 28 (2): 125–132 (2009).
9. Tunç, C. On the existence of periodic solutions of functional differential equations of third order. *Applied and Computational Mathematics* 15 (2): 189-199 (2016).
10. Vlcek, V. Note on existence of periodic solutions to the third-order nonlinear differential equation. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* 29 (1): 165-195 (1990).
11. Zhang, Z., Z. Wang & J. Yu, On the existence of periodic solutions of third order functional differential equations. *Funkcialaj Ekvacioj* 43 (3): 461–469 (2000).