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Research Article

Some Local-value Relationships Related to the Star Puzzle

Abdullah-Al-Kafi Majumdar*

Ritsumeikan Asia-Pacific University, 1–1 Jumonjibaru, Beppu-shi 874-8577, Japan

Abstract: The *star puzzle* is a variant of the classical Tower of Hanoi problem, where, in addition to the three pegs, S, P and D, there is a fourth one such that all disc movements are either to or from the fourth peg. Denoting by MS(n) the minimum number of moves required to solve the star puzzle, MS(n) satisfies the following recurrence relation:

$$MS(n) = \min_{1 \le k \le n-1} \left\{ 2MS(n-k) + 3^k - 1 \right\}.$$

This paper studies more closely the above recurrence relation and gives some new relationships, including some local-value relationships.

Keywords: Star puzzle, three-in-a-row puzzle, recurrence relation, local-value relationships

1. INTRODUCTION

The *star puzzle*, introduced by Stockmeyer [1], is as follows: Three pegs, S, P and D, are arranged in an equilateral triangle, and the fourth peg is at the center 0. Each disc movement must be either to or from 0, that is, direct moves of discs between any two of the pegs S, P and D are not allowed. Initially, the n discs, D_1, D_2, …, D_n, are placed on the *source peg*, S, in a tower (in small-on-large ordering) *standard position* (with the largest disc, D_n, at the bottom, the second largest disc, D_{n-1}, above it, and so on, with the smallest disc, D_1, at the top). The problem is to shift this tower of n discs from S to the *destination peg*, D, in minimum number of moves, using the *auxiliary peg* P, under the condition that each move can transfer only the topmost disc from one peg to another such that no disc is ever placed on top of a smaller one.

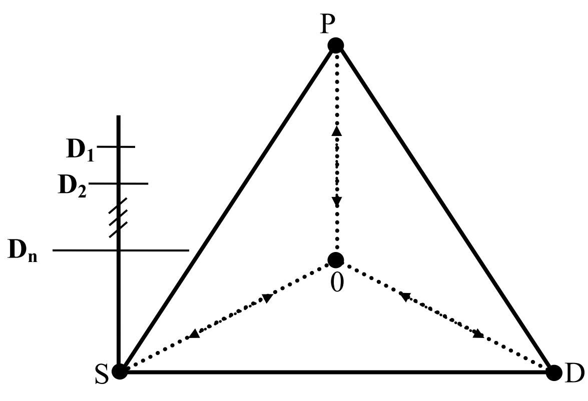

Fig. 1. The Star puzzle.

Let MS(n) denote the minimum number of moves required to transfer the tower of n discs from the source peg, S, to the destination peg, D (under the conditions of the problem). Then, MS(n) satisfies the following dynamic programming equation, due to Stockmeyer [1].

$$MS(n) = \min_{1 \le k \le n-1} \left\{ 2MS(n-k) + 3^k - 1 \right\} \; n \ge 2, \tag{1.1}$$

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*E-mail: aakmajumdar@gmail.com / majumdar@apu.ac.jp

with

$$MS(0) = 0,\ MS(1) = 2. \tag{1.2}$$

To transfer the tower of n discs from the source peg, S, to the destination peg, D, the scheme followed is:

Step 1: move the tower of topmost (smallest, consecutive) $n-k$ discs from the peg S to the peg P, using all the four pegs available.

The (minimum) number of moves required is $MS(n-k)$.

Step 2: transfer the remaining k (largest, consecutive) discs from the peg S to the peg D, using the three pegs available.

During this step, the pegs S, 0 and D (in this order) may be regarded as being arranged in a row (so that, we have the *three-in-a-row puzzle* with k discs), and the minimum number of moves required is 3^k-1.

Step 3: move the tower (of $n-k$ discs) from P on top of the discs on D (using all the four pegs available), again in (minimum) $MS(n-k)$ number of moves.

The total number of moves involved in the above three steps is

$$FS(n, k) \equiv 2MS(n-k) + 3^k - 1, \tag{1.3}$$

where k ($1 \le k \le n-1$) is to be determined such that FS(n, k) is minimized.

For details on the three-in-a-row puzzle, we refer to the paper of Scorer, Grundy and Smith [2] and Majumdar [3].

An equivalent form of (1.1) is the following

$$MS(n) = \min_{1 \le t \le n-1} \left\{ 2MS(t) + 3^{n-t} - 1 \right\},\ n \ge 2. \tag{1.4}$$

The following lemmas have been established by Majumdar [3].

Lemma 1.1 **:** MS(n) is an even (positive) integer for any integer $n \ge 1$.

Lemma 1.2 **:** MS(n) is not attained at two consecutive values of k.

Lemma 1.3 **:** For $n \ge 4$, MS(n) is not attained at $k = n-1$.

Lemma 1.4 **:** $MS(n+1) > MS(n)$, $n \ge 1$.

Lemma 1.5 **:** For any $n \ge 1$, $MS(n+2) - MS(n+1) \le 2\{MS(n+1) - MS(n)\}$.

This paper gives some local-value relationships related to the star puzzle, in Section 2 below.

2. SOME LOCAL-VALUE RELATIONSHIPS

We start with the following lemma.

Lemma 2.1 **:** Let MS(n) be attained at $t = t_1$ and $MS(n+1)$ be attained at $t = t_2$. Then, $t_2 \ge t_1$.

Proof **:** Let, on the contrary, $t_2 < t_1$. Now, since

$$MS(n) = 2MS(t_1) + 3^{n-t_1} - 1 \le 2MS(t_2) + 3^{n-t_2} - 1,$$

$$MS(n+1) = 2MS(t_2) + 3^{n-t_2+1} - 1 \le 2MS(t_1) + 3^{n-t_1+1} - 1,$$

we are led to the following chain of inequalities :

$$3^{n-t_2+1} - 3^{n-t_1+1} \le 2[MS(t_1) - MS(t_2)] \le 3^{n-t_2} - 3^{n-t_1},$$

which is absurd. This contradiction establishes that $t_2 \ge t_1$.

Corollary 2.1 **:** Let MS(n) be attained at the point $k = k_1$, and MS(n + 1) be attained at $k = k_2$. Then, $k_2 \le k_1 + 1$.

Proof **:** The proof follows from Lemma 2.1, since $t_1 = n - k_1$, $t_2 = n + 1 - k_2$.

Lemma 2.2 **:** Let MS(n) be attained at $k = k_1$. Then, MS(n) is not attained at $k = k_1 - 1$.
Proof **:** If MS(n) is attained at both $k = k_1$ and $k = k_1 - 1$, then

$$MS(n) = 2MS(n - k_1) + 3^{k_1} - 1 = 2MS(n - k_1 + 1) + 3^{k_1-1} - 1,$$

giving

$$MS(n - k_1 + 1) - MS(n - k_1) = 3^{k_1-1},$$

and we are led to a contradiction to Lemma 1.1.

Lemma 2.3 **:** For any integer $n \ge 1$,

(a) $MS(n+2) - MS(n+1) > MS(n+1) - MS(n)$,

(b) MS(n) is attained at a unique value of k.

Proof **:** The proof of part (a) is by induction on n. Since

$$MS(3) - MS(2) = 6 > 4 = MS(2) - MS(1),$$

we see that the result is true for $n = 1$. To proceed by induction, we assume that the result is true for some n.

First, we prove that MS(n) is attained at a unique value of k. Let MS(n) be attained at $k = k_1$. By Lemma 1.2, MS(n) is not attained at $k = k_1 + 1$. Therefore,

$$MS(n) = 2MS(n - k_1) + 3^{k_1} - 1 < 2MS(n - k_1 - 1) + 3^{k_1+1} - 1,$$

giving

$$MS(n - k_1) - MS(n - k_1 - 1) < 3^{k_1}. \tag{1}$$

Then, MS(n) is not attained at $k = k_1 + 2$, for otherwise

$$MS(n) = 2MS(n - k_1 - 2) + 3^{k_1+2} - 1 < 2MS(n - k_1 - 1) + 3^{k_1+1} - 1,$$

which gives

$$MS(n - k_1 - 1) - MS(n - k_1 - 2) > 3^{k_1+1}. \tag{2}$$

But then, (1) and (2), together with the induction hypothesis, give the following chain of inequalities :

$$3^{k_1+1} < MS(n - k_1 - 1) - MS(n - k_1 - 2)$$

$$< MS(n - k_1) - MS(n - k_1 - 1) < 3^{k_1},$$

which is absurd. Therefore, MS(n) is not attained at $k = k_1 + 2$ either. Continuing the argument, we see that MS(n) is not attained at any other values of k.

Now, let $MS(n+1)$ be attained at $k = k_2$. We want to show that $k_2 \geq k_1$. Let, on the contrary, $k_1 > k_2$. Now, since

$$MS(n) = 2MS(n-k_1) + 3^{k_1} - 1 < 2MS(n-k_2) + 3^{k_2} - 1,$$

$$MS(n+1) = 2MS(n-k_2+1) + 3^{k_2} - 1 < 2MS(n-k_1+1) + 3^{k_1} - 1,$$

we get the following chain of inequalities :

$$2[MS(n-k_2) - MS(n-k_1)] > 3^{k_1} - 3^{k_2} > 2[MS(n-k_2+1) - MS(n-k_1+1)],$$

which violates the induction hypothesis. Thus, $k_2 \geq k_1$. In fact, by virtue of Corollary 2.1, k_1 and k_2 satisfy the condition that $k_1 \leq k_2 \leq k_1 + 1$.

We now complete the proof of part (a) of the lemma.

Let $MS(n+2)$ be attained at $k = k_3$. Then, one of the following four cases may arise.

Case (A) : Let $k_1 = k_2 = k_3 = K$.
In this case,

$$MS(n+2) - MS(n+1) = 2[MS(n-K+2) - MS(n-K+1)],$$

$$MS(n+1) - MS(n) = 2[MS(n-K+1) - MS(n-K)],$$

and the result follows by virtue of the induction hypothesis.

Case (B) : Let $k_1 = k_2 = K, k_3 = K+1$.

Here,

$$MS(n+1) = 2MS(n-K+1) + 3^{K} - 1 < 2MS(n-K) + 3^{K+1} - 1,$$

$$MS(n+2) = 2MS(n-K+1) + 3^{K+1} - 1,$$

so that

$$MS(n+2) - MS(n+1) > 2[MS(n-K+1) - MS(n-K)] = MS(n+1) - MS(n).$$

Case (C) : Let $k_1 = K, k_2 = k_3 = K+1$.
In this case,

$$MS(n+2) = 2MS(n-K+1) + 3^{K+1} - 1,$$

$$MS(n+1) = 2MS(n-K) + 3^{K+1} - 1 < 2MS(n-K+1) + 3^{K} - 1,$$

so that

$$MS(n+2) - MS(n+1) > 2.3^{K} = MS(n+1) - MS(n),$$

and the result follows.

Case (D) : Let $k_1 = K, k_2 = K+1, k_3 = K+2$.
Here,

$$MS(n+1) - MS(n) = 2.3^{K}.$$

Now, if $MS(n+2)$ is attained at $k = K+2$, so that

$$MS(n+2) = 2MS(n-K) + 3^{K+2} - 1,$$

then

$$MS(n+2) - MS(n+1) = 2.3^{K+1} = 3[MS(n+1) - MS(n)],$$

which violates Lemma 1.5. Hence, this case cannot occur.

All these complete the proof of the lemma.

***Corollary 2.2* :** If, for some integer $n \geq 1$, MS(n) is attained at the point $k = k_1$ and MS(n + 1) is attained at $k = k_2$, then $k_1 \leq k_2 \leq k_1 + 1$.

***Corollary 2.3* :** If, for some integer $n \geq 1$, MS(n) is attained at the point $k = K$ and MS(n + 1) is attained at $k = K + 1$, then MS(n + 2) must be attained at $k = K + 1$.

From the computational point of view, Corollary 2.2 allows us to calculate recursively the value of k where MS(n + 1) is attained, starting with the value of k at which MS(n) is attained. Part (a) of Lemma 2.3 shows that, MS(n) is (strictly) convex in n in the sense of the inequality therein. It also shows that, MS(n+ 1) – MS(n) is strictly increasing in n.

***Lemma 2.4* :** Let, for some integer $n \geq 2$,

$$MS(n) - MS(n-1) = 2^s \text{ for some integer } s \geq 1. \tag{2.1}$$

Then, MS(n – 1) and MS(n) both are attained at the same value of k.
***Proof* :** Let MS(n) be attained at $k = k_1$, so that

$$MS(n) = 2MS(n - k_1) + 3^{k_1} - 1.$$

If MS(n – 1) is not attained at $k = k_1$, then it must be attained at $k = k_1 - 1$, so that

$$MS(n-1) = 2MS(n - k_1) + 3^{k_1 - 1} - 1.$$

But then,

$$MS(n) - MS(n-1) = 2.3^{k_1 - 1},$$

which violates the condition (2.1). This contradiction establishes the lemma.

Since

$$MS(1) - MS(0) = 2.$$

$$MS(2) - MS(1) = 2^2,$$

we see that such an n (satisfying the relationship (2.1)) exists. In this case,

$$MS(n-1) = 2MS(n - k_1 - 1) + 3^{k_1} - 1,$$

so that

$$MS(n) - MS(n-1) = 2[MS(n - k_1) - MS(n - k_1 - 1)] = 2^s,$$

which shows that, $MS(n - k_1 - 1)$ and $MS(n - k_1)$ both are attained at the same value of k.

Note that the converse of Lemma 2.4 is not true. Thus, if MS(n – 1) and MS(n) both are attained at the same value of k, then MS(n – 1) and MS(n) need not satisfy the condition (2.1). For example, MS(4) and MS(5) both are attained at $k = 2$, with

$$MS(5) - MS(4) = 12.$$

***Lemma 2.5* :** Let, for some integer $n \geq 1$,

$$MS(n) - MS(n-1) = 2.3^{\ell} \text{ for some integer } \ell \geq 0. \tag{2.2}$$

Let MS(n) be attained at $k = K$. Then, MS(n – 1) is attained at $k = K - 1$.
***Proof* :** By assumption,

$$MS(n) = 2MS(n - K) + 3^K - 1.$$

Then, MS(n – 1) must be attained at $k = K - 1$, for otherwise, it is attained at $k = K$, so that

$$MS(n-1) = 2MS(n-K-1) + 3^K - 1.$$

But then,

$$MS(n) - MS(n-1) = 2[MS(n-K) - MS(n-K-1)],$$

which violates the condition (2.2). This contradiction establishes the lemma.

In course of proving Lemma 2.5, we also proved the following

Corollary 2.4 **:** For some integer $n \geq 1$, $MS(n-1)$ and $MS(n)$ satisfy the relationship (2.2) if and only if $MS(n-1)$ and $MS(n)$ are attained at different values of k.

Corollary 2.5 **:** Let, for some integer $n \geq 1$, $MS(n-1)$ and $MS(n)$ satisfy the relationship (2.2). Let $MS(n)$ be attained at $k = K$. Then, $MS(n+1)$ is attained at $k = K$.

Proof **:** If $MS(n+1)$ is not attained at $k = K$, then it must be attained at $k = K+1$, so that

$$MS(n+1) = 2MS(n-K) + 3^{K+1} - 1,$$

which gives

$$MS(n+1) - MS(n) = 2.3^K,$$

so that

$$MS(n+1) - MS(n) = 2.3^K = 3[MS(n) - MS(n-1)],$$

violating Lemma 1.5. Therefore, $MS(n+1)$ is attained at $k = K$, so that

$$MS(n+1) = 2MS(n-K+1) + 3^K - 1 < 2MS(n-K) + 3^{K+1} - 1,$$

and hence,

$$MS(n+1) - MS(n) = 2[MS(n-K+1) - MS(n-K)] > 2.3^K.$$

Since

$$MS(1) - MS(0) = 2 = 2.3^0,$$

$$MS(3) - MS(2) = 6 = 2 \times 3,$$

we see that such an n (satisfying the relationship (2.2)) exists.

For any $n \geq 1$ fixed, let

$$FS(n,t) \equiv 2MS(t) + 3^{n-t} - 1; \quad 1 \leq t \leq n-1. \tag{2.3}$$

Lemma 2.6 **:** $FS(n, t)$ is strictly convex in t in the sense that

$$FS(n, t+2) - FS(n, t+1) > FS(n, t+1) - FS(n, t) \text{ for all } 0 \leq t \leq n-2.$$

Proof **:** Since

$$FS(n, t+2) - FS(n, t+1) = 2[MS(t+2) - MS(t+1)] - 2.3^{n-t-2},$$

$$FS(n, t+1) - FS(n, t) = 2[MS(t+1) - MS(t)] - 2.3^{n-t-1},$$

we get

$$[FS(n, t+2) - FS(n, t+1)] - [FS(n, t+1) - FS(n, t)]$$

$$= 2[\{MS(t+2) - MS(t+1)\} - \{MS(t+1) - MS(t)\}] + 4.3^{n-t-2}.$$

The result now follows by virtue of Lemma 2.3(a).

Lemma 2.7 **:** Let $N \geq 1$ be such that $MS(N-1)$ is attained at $k = K-1$ and $MS(N)$ is attained at $k = K$, so that

$$MS(N) - MS(N-1) = 2.3^{K-1}. \tag{2.4}$$

Then, there is an integer $M \geq 1$ such that

$$MS(N+M+1) - MS(N+M) = 2.3^K. \quad (2.5)$$

Proof **:** Since $MS(N+1)$ is attained at $k=K$, we get

$$MS(N+1) = 2MS(N+1-K) + 3^K - 1 < 2MS(N-K) + 3^{K+1} - 1,$$

so that

$$MS(N-K+1) - MS(N-K) < 3^K.$$

Continuing in this way, we see that

$$MS(N-K+m) - MS(N-K+m-1) < 3^K;\ m = 1, 2,$$

Since for $N \geq 1$ and $K \geq 1$ fixed, $\{MS(N-K+m) - MS(N-K+m-1)\}_{m=1}^{\infty}$ is strictly increasing in m, there is an integer $m \geq 1$ such that

$$MS(N+m-K+1) - MS(N+m-K) > 3^K.$$

For minimum such m, say, $m = M$, $MS(N+M)$ is attained at $k=K$, but $MS(N+M+1)$ is attained at $k=K+1$, so that

$$MS(N+M+1) - MS(N+M) = 2.3^K. \quad (2.6)$$

Thus, the lemma is established.

From the above proof, we see that, for all m with $1 \leq m \leq M$,

$$2.3^{K-1} < MS(N+m) - MS(N+m-1)$$

$$= 2[MS(N-K+m) - MS(N-K+m-1)] < 2.3^K. \quad (2.7)$$

Now, since

$$MS(N+M+1) = 2MS(N+M-K) + 3^{K+1} - 1 < 2MS(N+M-K+1) + 3^K - 1,$$

we see that

$$MS(N+M-K+1) - MS(N+M-K) > 3^K > MS(N) - MS(N-1).$$

Therefore,

$$N+M-K+1 > N,$$

that is,

$$M \geq K.$$

Let a_n be defined by

$$a_n = MS(n) - MS(n-1),\ n \geq 1. \quad (2.8)$$

Let $m_j \geq 1$ be the integer, defined as follows :

$$a_{m_j} = MS(m_j) - MS(m_j - 1) = 2.3^j;\ j \geq 0,$$

with

$$m_0 = 1,\ m_1 = 3.$$

Then, $MS(m_j - 1)$ is attained at $k=j$, and for all n with $m_j \leq n \leq m_{j+1} - 1$, $MS(n)$ is attained at $k = j+1$.

Let N be such that

$$MS(N+1) - MS(N) = 2^s \text{ for some integer } s \geq 1. \quad (2.9)$$

Then, $MS(N)$ and $MS(N+1)$ both must be attained at the same value of k, say, $k=K$, so that

$$MS(N) = 2MS(N-K) + 3^K - 1 < 2MS(N-K+1) + 3^{K-1} - 1,$$

$$MS(N+1) = 2MS(N-K+1) + 3^K - 1 < 2MS(N-K) + 3^{K+1} - 1,$$

and we get the following chain of inequalities :

$$2.3^{K-1} < MS(N+1) - MS(N)$$
$$= 2[MS(N-K+1) - MS(N-K)] < 2.3^K. \quad (3)$$

Let MS(M) and MS(M + 1) both be attained at k = L. Then,

$$MS(M+1) = 2MS(M-L+1) + 3^L - 1,$$
$$MS(M) = 2MS(M-L) + 3^L - 1,$$

so that

$$MS(M+1) - MS(M) = 2[MS(M-L+1) - MS(M-L)]. \quad (2.10)$$

Letting

$$M - L = N, \quad (4)$$

we see that the next pair of functions is MS(M) and MS(M + 1), with

$$MS(M+1) - MS(M) = 2^{s+1}.$$

Now, since

$$2.3^{L-1} < MS(M+1) - MS(M) < 2.3^L, \quad (5)$$

from (3) and (5), we must have

$$4.3^K > 2.3^{L-1},\ 2.3^L > 4.3^{K-1},$$

so that

$$K \le L \le K+1.$$

From (3), we may estimate s of (2.9) as follows : Since

$$3^{K-1} < 2^{s-1} < 3^K,$$

we get

$$\frac{\ln 3}{\ln 2}(K-1) + 1 < s < \frac{\ln 3}{\ln 2}K + 1. \quad (2.11)$$

Thus, for example, if K = 1, the only s satisfying the inequality (2.11) is s = 2, while for K = 2, there are two values of s satisfying (2.11), namely, s = 3, 4.

Let the integers $k_j \ge 1$ be defined as follows :

$$a_{k_j} = MS(k_j) - MS(k_j - 1) = 2.2^j;\ j \ge 0,$$

with

$$k_0 = 1,\ k_1 = 2.$$

Lemma 2.8 **:** For all $j \ge 1$, $MS(k_j)$ is attained at $k = k_j - k_{j-1}$.

Proof **:** The proof is by induction on j. Since $MS(k_1) = MS(2)$ is attained at $k = k_1 - k_0 = 2 - 1 = 1$, we see that the result is true for j = 1. So, we assume that the result is true for some j, that is, we assume that $MS(N+1) = MS(k_j)$ is attained at $k = k_j - k_{j-1}$.
Now, the above analysis shows that $MS(M+1) = MS(k_j + 1)$ is attained at k = L. From (4),

$$L = M - N = k_{j+1} - k_j.$$

Thus, the result is true for j + 1 as well, completing induction.

Let $\{b_n\}_{n=1}^{\infty}$ be the sequence of integers in increasing order :

$$b_n = 2^i 3^\ell,\ i \ge 0,\ \ell \ge 0, \quad (2.12)$$

(so that, $\{b_n\}$ is the sequence of numbers $\{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, \ldots\}$). Then, we have the following result, due to Matsuura [4].

Lemma 2.9 **:** Let n be such that $2^i < b_n < 2^{i+1}$ for some integer $i \geq 1$. Then,

$$b_n = 3b_{n-i-1}.$$

Corollary 2.6 **:** Let c_n be defined as follows :

$$c_n = 2b_n,\ n \geq 1.$$

Let $2^i < b_n < 2^{i+1}$ for some integer $i \geq 1$. Then,

$$c_n = 3c_{n-i-1}.$$

Given any integer $K \geq 1$, a related problem of interest is to find the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ such that

$$3^K < 2^i 3^\ell < 3^{K+1}. \tag{6}$$

Now, since the above inequality reduces to

$$3^{K-\ell} < 2^i < 3^{K-\ell+1},$$

we see that i must satisfy the following inequality :

$$(K-\ell)\frac{\ln 3}{\ln 2} < i < (K-\ell+1)\frac{\ln 3}{\ln 2}. \tag{7}$$

Thus, for example, with $K=1$ in (7), for $\ell=0$, we have $i=2, 3$; and for $\ell=1$, we have $i=1$; that is, there are exactly three elements of the sequence $\{b_n\}_{n=1}^{\infty}$ satisfying the inequality

$$3 < 2^i 3^\ell < 3^2,$$

namely, 2^2, 2^3 and 2×3.

Let $N(n, K)$ denote the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ satisfying the inequality (6), that is,

$$N(n, K) = \left|\left\{b_n : 3^K < b_n < 3^{K+1}\right\}\right| = \left|\left\{b_n : 3^K < 2^i 3^\ell < 3^{K+1}\right\}\right|.$$

Then, we have the following lemma.

Lemma 2.10 **:** For any integer $K \geq 1$,

$$N(n, K+1) = N(n, K) + \left|\left\{i : 3^{K+1} < 2^i < 3^{K+2}\right\}\right|.$$

Proof **:** From (7), we note that, for K + 1 with ℓ = 1, 2, ..., K + 1 corresponds to the case K with ℓ = 0, 1, ..., K. Hence, the result follows.

Lemma 2.11 **:** For any integer $K \geq 1$,

(a) $\left|\left\{b_n : 3^K < b_n < 3^{K+1}\right\}\right| = \max\left\{i : 2^i < 3^{K+1}\right\}$,

(b) $\left|\left\{b_n : 3^K < b_n < 3^{K+1}\right\}\right| = \left|\left\{c_n : 2.3^K < c_n < 2.3^{K+1}\right\}\right|$.

Proof **:** Part (a) follows from Lemma 2.10 by induction on K. The proof of part (b) is evident from part (a), by virtue of Corollary 2.6.

3. DISCUSSION

Stockmeyer [1] gave a sketch of the proof that MS(n) can be expressed as

$$MS(n) = \sum_{m=1}^{n} a_m = 2\sum_{m=1}^{n} b_m .$$

but his argument is rather heuristic in nature, and is not supported by any theoretical development. In particular, the following points remain to be resolved :

1. There is a one-to-one correspondence between the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$,
2. The sequences $\{3^{n-1}\}_{n=1}^{\infty}$ and $\{2b_n\}_{n=1}^{\infty}$ exactly partition the sequence $\{b_n\}_{n=1}^{\infty}$, as has been claimed by Stockmeyer [1] without any proof.

This paper gives an exact analysis of the recurrence relation (1.1), which reveals some interesting properties. For example, if $m_j \le n \le m_{j+1} - 1$, MS(n) is attained at the unique point $k = j + 1$, a result which justifies the claim of Stockmeyer [1].

Corollary 2.2 may be exploited to find MS(n) recursively. In Table 1, we give the values of MS(n) for $n = 13(1)24$ to supplement Table given in Majumdar [3].

Table 1. Values of MS(n) and k=k(n) for n=13(1)24.

n	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
MS(n)	324	396	492	600	728	872	1034	1226	1442	1698	1986	2310
k	4	4	4	4	4	4	5	5	5	5	5	5

In a recent paper, Majumdar [4] has treated in detail the following recurrence relation, due to Matsuura [5].

$$T(n,\alpha) = \min_{0 \le k \le n-1} \{\alpha T(k,\alpha) + S(n-k,3)\},$$

where $S(n,3) = 2^n - 1$ is the solution of the Tower of Hanoi problem. It is an interesting problem to show that $MS(n) = 2T(n,3)$ for all $n \ge 1$, directly from the corresponding optimality equations.

4. REFERENCES

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