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Research Article

On Classes of Analytic Functions Associated by a Parametric Linear Operator in a Complex Domain

Rabha W. Ibrahim[1], Adem Kilicman[2], and Zainab E. Abdulnaby[2,3,*]

[1]Faculty of Computer Science and Information Technology, University of Malaya, 50603, Malaysia
[2]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia,
43400 UPM Serdang, Selangor, Malaysia
[3]Department of Mathematics, College of Science, Al-Mustansiriyah University, Baghdad, Iraq

Abstract: The present paper deals with some geometric classes of analytic functions, such as starlike, convex and bounded turning property in a complex domain. These classes are defined by a new linear operator in the normalized space of analytic functions. The linear operator is introduced by a convolution of Szego function involving parametric coefficients type *Laguerre* polynomial, with the normalized function. Sufficient conditions on this operator are illustrated to study the geometric properties. Our tool is based on some recent results in this direction. The main strategy for this work is to provide parametric functional inequalities in the open unit disk.

Keywords: analytic functions, univalent function, starlike function; convex function, bounded turning function, *Laguerre* polynomial, convolution (or Hadamard product), Riemann zeta function

1. INTRODUCTION

Special function is a major topic in classical analysis which deals with class of mathematical functions and that may arise in the solution of various classical problems of some branch of mathematics and mathematical physics. In particular, most special functions are considered as a function of a complex variable. An intricate special function can be expressed in terms of simpler function and the simplest way to assess a function is to expand it by a Taylor series.

Further, special functions have proven their eligibility in associated with an analytic function via convolution technique (or Hadamard product) to define, prove, represent and extend several types of operators, here we suggest a number of the well-known and recent linear operators defined according to special function (see [1-5]). Hohlov [6] derived sufficient conditions that guarantee such mappings for the operator defined by means of the Hadamard product with the Gauss hypergeometric function. Carlson and Shaffer [7] adopted the incomplete beta function to define a linear operator, which has been widely used in the space of analytic and univalent functions in the open unit disk $\mathbb{U}$ (see [8]). After more decades Dzoik and Srivastava [9] investigated a convolution linear operator which has been formulated in terms of the generalized hypergeometric function. Several interesting properties and characteristics of the Dziok–Srivastava operator are derived (see [10], [20]). Following, by interesting related work, Srivastava [11] determined also, another extension covering the Wright function which is known as Srivastava-Wright operator (see [12]). Recently, in Srivastava and Attiya [13] defined an integral operator in terms of convolution with Hurwitz–Lerch Zeta function and studied some differential subordination results associated with the operator. It is worth mentioning that, Srivastava-Attiya operator is a generalized of many operators.

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*Corresponding author, Email address: esazainab@yahoo.com

Ibrahim [14] introduced a linear operator for analytic functions derived by using the basic hypergeometric series, also investigated some applications of differential subordinate on Jack's lemma.

The paper is organized as follows:Section 2 revisesa new linear operator in the normalized space of analytic functions is defined .The linear operator is introduced by a convolution of Szego function involving parametric coefficients type *Laguerre* polynomial, with the normalized function. Further, some geometric classes of analytic functions, such as starlike, convex and bounded turning property in a complex domain are investigated in Section 3. These classes are sufficient conditions on this operator are illustrated to study the geometric properties. Our method is based on some recent results in this direction. The main strategy of this work is to provide parametric functional inequalities in the open unit disk.

2. PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + a_2 z^2 + a_3 z^3 \cdots, \qquad a_1 = 1 \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C}: \ |z| < 1\}$ normalized by $f(0) = 1 - f^{'}(0) = 0,$ and let $\mathcal{S}$ be the subclass of the $\mathcal{A}$ of the univalent functions in $\mathbb{U}$. Further, a function $f(z) \in \mathcal{S}$ is said to be starlike and convex, if their geometric condition satisfies

$$\Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > 0 \quad \text{and} \quad \Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > 0,$$

respectively, these subclasses of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and K. The convolution (or Hadamard product), between two analytic functions, function of the form (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ in $\mathbb{U},$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k\, z^k = (g * f)(z). \tag{2}$$

A function of bounded turning $\mathcal{B}$ if it satisfies the inequality

$$\Re\{f'(z)\} > 0, \qquad z \in \mathbb{U}.$$

We begin our current consideration by recalling that a generating function of the associated *Laguerre* polynomials $G(b, \tau, z)$ defined by Szego [15].

$$G(b, \tau, z) = (1 - z)^{-b-1} e^{(-\tau\, z/1-z)} = \sum_{k=0}^{\infty} L_k^{(b)}(\tau) z^k, \qquad z \in \mathbb{U} \tag{3}$$

$$(b \in \mathbb{C} \setminus \{-1, -2, \ldots\};\ \Re(b) > 1;\ \tau \in \mathbb{R};\ |z| < 1)$$

where $L_k^{(b)}(\tau)$ is the form of generalized Laguerre polynomials of degree k on the interval $(0, \infty)$, is given by

$$L_k^{(b)}(\tau) = \sum_{i=0}^{k} \binom{k+b}{k-i} \frac{(-\tau)^i}{i!}, \quad k = 0,1, \ldots$$

$$= b + \binom{1+b}{1} - \binom{1+b}{0}\tau + \binom{2+b}{2} - \binom{2+b}{1}\tau + \binom{2+b}{0}\frac{\tau^2}{2!} + \cdots \tag{4}$$

where $\binom{\propto}{\mu}$ is the generalized binomial coefficient

$$\binom{\propto}{\mu} = \frac{\Gamma(\propto+1)}{\Gamma(\mu+1)\Gamma(\propto-\mu+1)}.$$

Now, for $f \in \mathcal{A},\ b \in \mathbb{C}\setminus\{-1,-2,\ldots\}$ and $\tau \in \mathbb{R}$, let define the function $\Lambda(b,\tau,z)$ by

$$\Lambda(b,\tau,z) = \frac{G(b,\ \tau,\ z)}{L_1^{(b)}(\tau)}, \qquad (z \in \mathbb{U}) \tag{5}$$

where

$$L_1^{(b)}(\tau) = \binom{1+b}{1} - \binom{1+b}{0}\tau \neq 0,$$

We proceed to define a linear operator $\mathcal{G}_\tau^b : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{G}_\tau^b f(z) = \Lambda(b,\tau,z) * f(z), \tag{6}$$

$$(f \in \mathcal{A};\ z \in \mathbb{U};\ \tau \in \mathbb{R};\ b \in \mathbb{C}\setminus\{-1,-2,\ldots\})$$

by the following Hadamard product (2), we obtain

$$\mathcal{G}_\tau^b f(z) = z + \sum_{k=2}^{\infty} \Omega_{b,k}(\tau) a_k z^k \qquad (z \in \mathbb{U}), \tag{7}$$

where

$$\Omega_{b,k}(\tau) = \frac{L_k^{(b)}(\tau)}{L_1^{(b)}(\tau)}, \qquad L_1^{(b)}(\tau) \neq 0. \tag{8}$$

Remark 2.1: From (7) and (8), we have

$$\mathcal{G}_\tau^0 f(z) = \lim_{b\to 0}\{\mathcal{G}_\tau^b f(z)\}. \tag{9}$$

It is clear that

$$\mathcal{G}_0^0 f(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z) \quad and \qquad \mathcal{G}_0^{\alpha-1} f(z) = z(1-z)^{-\alpha}.$$

Using the relation (7), we obtain

$$z\left(\mathcal{G}_\tau^{b+1} f(z)\right)' = b\,\mathcal{G}_\tau^b f(z) - (b-1)\mathcal{G}_\tau^{b+1} f(z),$$

and

$$\mathcal{G}_\tau^b\big(z f'(z)\big) = z\left(\mathcal{G}_\tau^b f(z)\right)'. \tag{10}$$

Our aim is to study the operator (7) in view of the geometric function theory, by introducing some conditions on $\Omega_{b,k}(\tau)$. For this purpose we need, the following results.

Theorem 2.2: (Nunokawa [16]) . If $f(z)$ is analytic function in $\mathcal{A}$, satisfies $|f''(z)| < 1,\ (z \in \mathbb{U})$, then $f(z) \in \mathcal{S}$.

Theorem 2.3: (Mocanu [17]) . If $f(z)$ is analytic function defined by (1) and satisfies

$$|f'(z) - 1| < \frac{\sqrt{20}}{5}(z \in \mathbb{U}), \tag{11}$$

then $f(z)$ belongs to $\mathcal{S}$.

3. RESULTS

In this section, we concentrate on some results for coefficient of $f(z)$ to be in the class $\mathcal{S}$ and for related classes.

Theorem 3.1: Let $0 \le \tau < 1$ and $\Re(b) > 1$. If $\mathcal{G}_\tau^b f(z) \in \mathcal{A}$ such that

$$|a_k| \le \frac{b-1}{k^{2(1+b)}}, \qquad k \ge 2.$$

Then $f(z) \in \mathcal{S}$.

Proof: Our aim to apply Theorem 2.2. It is well-known that, when $0 \le \tau < 1$, the polynomial $L_k^{(b)}(\tau)$ takes its maximality at $\tau = 0$ for all $k \ge 2$ and $\Re(b) > 1$, such that

$$L_k^b(0) = \frac{k^b}{\Gamma(b+1)}.$$

Consequently, we obtain

$$\left|\Omega_{b,k}(\tau)\right| \le k^b, \qquad \Re(b) > 1.$$

By the definition of $\mathcal{G}_\tau^b f(z)$, we have the following assertion:

$$\begin{aligned}
\left| [\mathcal{G}_\tau^b f(z)]'' \right| &\le \sum_{k=2}^{\infty} k(k-1)|a_k|\left|\Omega_{b,k}(\tau)\right| \\
&\le \sum_{k=2}^{\infty} k^{2+b}|a_k| \\
&\le (b-1)\left[1 + \sum_{k=2}^{\infty}\frac{1}{k^b}\right] \\
&\coloneqq (b-1)\,\zeta(b), \ \ \Re(b) \to 1,
\end{aligned}$$

where $\zeta(b)$ is denoted the Riemann zeta function $\left(\zeta(\varsigma) = \sum_{k=1}^{\infty}\frac{1}{k^\varsigma},\right)$ (see [18]). But

$$\lim_{b\to 1}(b-1)\,\zeta(b) = 1.$$

Thus, $\left| [\mathcal{G}_\tau^b f(z)]'' \right| < 1$, i.e. $\mathcal{G}_\tau^b f(z) \in \mathcal{S}$.

Theorem 3.2: Let $0 \le \tau < 1$ and $\Re(b) > 1$. If $\mathcal{G}_\tau^b f(z) \in \mathcal{A}$ such that

$$|a_k| \le \frac{\zeta(k) - 1}{k^{3+b}}, \qquad k \ge 2$$

where $\zeta(k) > 1$ is denoted the Riemann zeta function, then $\mathcal{G}_\tau^b f(z) \in \mathcal{S}$.

Proof: Since

$$\left| [\mathcal{G}_\tau^b f(z)]'' \right| \leq \sum_{k=2}^{\infty} k(k-1)|a_k|\left|\Omega_{b,k}(\tau)\right|$$

$$\leq \sum_{k=2}^{\infty} k^{2+b}|a_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k} = 1-\delta < 1,$$

where $\delta = 0.57721 \ldots$ is Euler's constant. Hence, this indicates that $\mathcal{G}_\tau^b f(z) \in \mathcal{S}$.

Theorem 3.3: Let $0 \leq \tau < 1$ and $\Re(b) > 1$. If $\mathcal{G}_\tau^b f(z) \in \mathcal{A}$ such that

$$|a_k| \leq \frac{\zeta(k)-1}{k^{2+b}}, \qquad k \geq 2, \quad \zeta(k) > 1,$$

where $\zeta(k)$ is denoted the Riemann zeta function, then $\mathcal{G}_\tau^b f(z) \in \mathcal{S}$. Moreover, $\mathcal{G}_\tau^b f(z) \in \mathcal{B}$.

Proof: Since

$$\left| [\mathcal{G}_\tau^b f(z)]'' \right| \leq \sum_{k=2}^{\infty} k(k-1)|a_k|\left|\Omega_{b,k}(\tau)\right|$$

$$\leq \sum_{k=2}^{\infty} k^{2+b}|a_k|$$

$$\leq \sum_{k=2}^{\infty} \zeta(k)-1 = 1,$$

Hence, this refers to $\mathcal{G}_\tau^b f(z) \in \mathcal{S}$. To prove that $\mathcal{G}_\tau^b f(z) \in \mathcal{B}$ it is sufficient to show that $\left| [\mathcal{G}_\tau^b f(z)]' \right| > 0$.

$$\left| [\mathcal{G}_\tau^b f(z)]' \right| = 1 + \sum_{k=2}^{\infty} k\, |a_k|\left|\Omega_{b,k}(\tau) z^{k-1}\right|$$

$$\geq 1 - \sum_{k=2}^{\infty} k^{1+b}|a_k|$$

$$\geq 1 - \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}$$

$$= \delta > 0.$$

This completes the proof.

Theorem 3.4: Let $0 \leq \tau < 1$ and $\Re(b) > 1$. If $\mathcal{G}_\tau^b f(z) \in \mathcal{A}$ such that

$$|a_k| \leq \frac{\zeta(2k)-1}{2^{2k} k^{2+b}}, \qquad k \geq 2, \quad \zeta(k) > 1,$$

where $\zeta(2k)$ is refereed the Riemann zeta function, then $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}$.

Proof: Since

$$\left| [\mathcal{G}_\tau^b\, f(z)]'' \right| \le \sum_{k=2}^{\infty} k(k-1)|a_k|\left|\Omega_{b,k}(\tau)\right|$$

$$\le \sum_{k=2}^{\infty} k^{2+b}|a_k|$$

$$\le \sum_{k=2}^{\infty} \frac{\zeta(2k)-1}{2^{2k}} = \frac{1}{6} < 1,$$

Hence, this refers to $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}$.

Theorem 3.5: Let $0 \le \tau < 1$ and $\Re(b) > 1$. If $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A}$ such that

$$|a_k| \le \frac{\zeta(2k-1)-1}{(k-1)k^{1+b}}, \qquad k \ge 2, \quad \zeta(k) > 1,$$

where $\zeta(2k)$ is refereed the Riemann zeta function, then $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}$.

Proof: Our aim is to apply Theorem 2.3. Since

$$\left|\sum_{k=2}^{\infty} k\, a_k \Omega_{b,k}(\tau)\right| \le \sum_{k=2}^{\infty} k\, |a_k|\left|\Omega_{b,k}(\tau)\right|$$

$$\le \sum_{k=2}^{\infty} k^{1+b}|a_k|$$

$$\le \sum_{k=2}^{\infty} \frac{\zeta(2k)-1}{(k-1)}.$$

Consequently, we obtain

$$|\, [\mathcal{G}_\tau^b\, f(z)]' - 1| \le \sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k} = \log 2 = 0.30102 \ldots < \frac{\sqrt{20}}{5} = 0.8944.$$

Thus, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}$.

Next, we illustrate some geometric properties of the operator $\mathcal{G}_\tau^b\, f(z)$ to be starlike and convex. For this purpose, we recall that the function belongs to $\mathcal{S}^*(\beta)$, $0 < \beta < 1$, if it achieves the inequality (see [19]).

$$\left|\frac{f(z)}{z\, f'(z)} - \frac{1}{2\,\beta}\right| < \frac{1}{2\,\beta}.$$

Moreover, the function$f$ is in the class $\kappa(\beta)$, $0 < \beta < 1$ if and only if $z\, f' \in \mathcal{S}^*(\beta)$. We have the following results:

Theorem 3.6: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A}$, $\tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{1}{2}\right)$ and

$$\sum_{k=2}^{\infty} k^{b+1}|a_k| \le \sum_{k=2}^{\infty} \frac{1-2\,\beta}{1+2\,\beta}$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$.

Proof: By the assumption of the theorem, we have the following conclusion:

$$\left|\frac{[\mathcal{G}_\tau^b f(z)]}{z\,[\mathcal{G}_\tau^b f(z)]'} - \frac{1}{2\,\beta}\right| \le \left|\frac{[\mathcal{G}_\tau^b f(z)]}{z[\mathcal{G}_\tau^b f(z)]'}\right|$$

$$\le \frac{1+\sum_{k=2}^{\infty} k^b |a_k|}{1-\sum_{k=2}^{\infty} k^{b+1}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty} k^{b+1} |a_k|}{1-\sum_{k=2}^{\infty} k^{b+1}|a_k|}$$

$$\le \frac{1}{2\,\beta}.$$

This yields that $\mathcal{G}_\tau^b f(z) \in \mathcal{S}^*,\ 0 < \beta < 1/2$.

Theorem 3.7: Let $\mathcal{G}_\tau^b f(z) \in \mathcal{A},\ \tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{1}{2}\right)$ and

$$\sum_{k=2}^{\infty} k^{b+2}|a_k| \le \sum_{k=2}^{\infty} \frac{1-2\,\beta}{1+2\,\beta}$$

then $\mathcal{G}_\tau^b f(z) \in \mathcal{K}(\beta)$.

Proof: By the assumption of the theorem, we have the following assertion:

$$\left|\frac{z\,[\mathcal{G}_\tau^b f(z)]}{z\left(z\,[\mathcal{G}_\tau^b f(z)]'\right)'} - \frac{1}{2\,\beta}\right| \le \left|\frac{z\,[\mathcal{G}_\tau^b f(z)]}{z\left(z\,[\mathcal{G}_\tau^b f(z)]'\right)'}\right|$$

$$\le \frac{1+\sum_{k=2}^{\infty} k^{b+1} |a_k|}{1-\sum_{k=2}^{\infty} k^{b+2}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty} k^{b+2} |a_k|}{1-\sum_{k=2}^{\infty} k^{b+2}|a_k|}$$

$$\le \frac{1}{2\,\beta}$$

This implies that $\mathcal{G}_\tau^b f(z) \in \mathcal{K}(\beta\,),\quad 0 < \beta < 1/2$.

Theorem 3.8: Let $\mathcal{G}_\tau^b f(z) \in \mathcal{A},\ \tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{5}{6}\right)$ and

$$|a_k| \le \frac{1}{k^{b+1}}$$

then $\mathcal{G}_\tau^b f(z) \in \mathcal{S}^*(\beta)$.

Proof: To show that the operator $\mathcal{G}_\tau^b f(z) \in \mathcal{S}^*(\beta)$, we conclude

$$\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'}-\frac{1}{2\,\beta}\right| \le \frac{1+\sum_{k=2}^{\infty} k^{b+1}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+1}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^0}}{1-\sum_{k=2}^{\infty}\frac{1}{k^0}} \le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^0}}{2-\sum_{k=1}^{\infty}\frac{1}{k^0}}$$

$$\le \frac{2+\zeta(0)}{2-\zeta(0)} = \frac{3/2}{5/2} = \frac{3}{5}, \quad \zeta(0) = -\frac{1}{2}$$

$$\le \frac{1}{2\,\beta}.$$

This yields that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta), \quad 0 < \beta < 5/6.$

Theorem 3.9: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A},\ \tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{5}{6}\right)$ and

$$|a_k| \le \frac{1}{k^{b+2}}$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,)$.

Proof: To prove that the operator $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,)$, we conclude

$$\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'}-\frac{1}{2\,\beta}\right| \le \frac{1+\sum_{k=2}^{\infty} k^{b+2}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+2}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^0}}{1-\sum_{k=2}^{\infty}\frac{1}{k^0}} \le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^0}}{2-\sum_{k=1}^{\infty}\frac{1}{k^0}}$$

$$\le \frac{2+\zeta(0)}{2-\zeta(0)} = \frac{3/2}{5/2} = \frac{3}{5}, \quad \zeta(0) = -\frac{1}{2}$$

$$\le \frac{1}{2\,\beta}.$$

This gives that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,), \quad 0 < \beta < 5/6.$

Theorem 3.10: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A},\ \tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{12-\pi^2}{2\,(12+\pi^2)}\right) \approx (0, 0.05)$ and

$$|a_k| \le \frac{1}{k^{b+3}},$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$.

Proof: To demonstrate that the operator $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$, we conclude

$$\begin{aligned}\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'} - \frac{1}{2\,\beta}\right| &\le \frac{1+\sum_{k=2}^{\infty} k^{b+1}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+1}|a_k|} \\ &\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^2}}{1-\sum_{k=2}^{\infty}\frac{1}{k^2}} \\ &\le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^2}}{2-\sum_{k=1}^{\infty}\frac{1}{k^2}} \\ &\le \frac{2+\zeta(2)}{2-\zeta(2)}, \quad \zeta(2)=\frac{\pi^2}{6} \\ &\le \frac{1}{2\,\beta}\end{aligned}$$

This gives that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$, $0 < \beta < 0.05$.

Theorem 3.11: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A}$, $\tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{12-\pi^2}{2\,(12+\pi^2)}\right) \approx (0, 0.05)$ and

$$|a_k| \le \frac{1}{k^{b+4}},$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,)$.

Proof: To show that the operator $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,)$, we conclude

$$\begin{aligned}\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'} - \frac{1}{2\,\beta}\right| &\le \frac{1+\sum_{k=2}^{\infty} k^{b+2}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+2}|a_k|} \\ &\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^2}}{1-\sum_{k=2}^{\infty}\frac{1}{k^2}} \\ &\le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^2}}{2-\sum_{k=1}^{\infty}\frac{1}{k^2}} \\ &\le \frac{2+\zeta(2)}{2-\zeta(2)}, \quad \zeta(2)=\frac{\pi^2}{6} \\ &\le \frac{1}{2\,\beta}.\end{aligned}$$

This gives that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,)$, $0 < \beta < 0.05$.

Theorem 3.12: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A}$, $\tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{1}{8}\right)$ and

$$|a_k| \le \frac{1}{k^{b+4}},$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$.

Proof: To show that the operator $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$, we conclude

$$\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'} - \frac{1}{2\,\beta}\right| \le \frac{1+\sum_{k=2}^{\infty} k^{b+1}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+1}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^3}}{1-\sum_{k=2}^{\infty}\frac{1}{k^3}}$$

$$\le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^3}}{2-\sum_{k=1}^{\infty}\frac{1}{k^3}}$$

$$\le \frac{2+\zeta(3)}{2-\zeta(3)}, \quad \zeta(3) = 1.2$$

$$\le \frac{1}{2\,\beta}.$$

This gives that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta), \quad 0 < \beta < 1/8.$

Theorem 3.13: Let $\mathcal{G}_\tau^b\, f(z) \in \mathcal{A},\ \tau \in (0,1)$ and $\Re(b) > 1$. If $\beta \in \left(0, \frac{1}{8}\right)$ and

$$|a_k| \le \frac{1}{k^{b+5}},$$

Then, $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,).$

Proof: To show that the operator $\mathcal{G}_\tau^b\, f(z) \in \mathcal{S}^*(\beta)$, we conclude

$$\left|\frac{z\,[\mathcal{G}_\tau^b\, f(z)]}{z\left(z\left[\mathcal{G}_\tau^b\, f(z)\right]'\right)'} - \frac{1}{2\,\beta}\right| \le \frac{1+\sum_{k=2}^{\infty} k^{b+2}|a_k|}{1-\sum_{k=2}^{\infty} k^{b+2}|a_k|}$$

$$\le \frac{1+\sum_{k=2}^{\infty}\frac{1}{k^3}}{1-\sum_{k=2}^{\infty}\frac{1}{k^3}}$$

$$\le \frac{2+\sum_{k=1}^{\infty}\frac{1}{k^3}}{2-\sum_{k=1}^{\infty}\frac{1}{k^3}}$$

$$\le \frac{2+\zeta(3)}{2-\zeta(3)}, \quad \zeta(3) = 1.2$$

$$\le \frac{1}{2\,\beta}.$$

This gives that $\mathcal{G}_\tau^b\, f(z) \in \mathcal{K}(\beta\,), \quad 0 < \beta < 1/8.$

4. CONCLUSIONS

We defined a new linear operator in terms of the generating function of the *Laguerre* polynomials. The method is concluded by Hadamard product. Based on this operator, we defined new classes of parametric coefficients in the open unit disk. Different studies have been illustrated, involving the geometric properties of these classes. We conclude that the best upper bound of these classes is determined by the Riemann zeta function.

5. REFERENCES

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