Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 55 (4): 71–77 (2018)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

Some Properties of Harmonic Univalent Functions in a Conic Domain

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Abstract: We investigate a new subclass $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,k,t)$of harmonic functions satisfying condition:

$$\Re\left\{\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}\right\}>k\left|\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}-1\right|+1-|\gamma|(z\in \mathrm{E}),$$

Where $\mathfrak{L}(z)=z^t-\sum_{j=2}^{\infty}|a_j|z^{j+t-1}$ and $\mathcal{K}(z)=\sum_{j=1}^{\infty}|b_j|z^{j+t-1}$, $|b_1|<1$ .We also determine the coefficients inequalities, growth and distortion bounds, radius of star likeness for the analytic part of the harmonic functions $\mathcal{F}(z)=\mathfrak{L}(z)+\mathcal{K}(z)$. For specific values of parameters involved, our findings may be related to the previously known results.

Keywords: Harmonic and univalent functions, Coefficients inequalities, Radius of starlikeness

1. INTRODUCTION

Harmonic functions are important because of their applications in minimal surfaces and these functions play a vital role in applied mathematics for example, [3, 5, and 6]. Harmonic functions have close connections with conformal mappings are not analytic in general and hence the Cauchy-Riemann equations do not hold. These functions were first studied by differential geometers and then complex analysts involved in their study which was initiated by Clunie and Sheil-Small [4] in 1984.

A continuous function $f(z)=u(x,y)+iv(x,y)$ is harmonic, if both u and v are harmonic. A harmonic function f takes the canonical form: $f(z)=l(z)+k(z)$, where l and k are analytic in $\mathbb{U}$. The characterization of f for local univalence and sense-preserving is just $|l'(z)|>|k'(z)|$ in $\mathbb{U}$. For detail, we refer [4,7]. Let $\mathcal{H}$ be the class of functions $f(z)=l(z)+k(z)$ univalent and sense-preserving in $\mathbb{U}$ such that $f(0)=f_z(0)-1=0$. A function $f\in\mathcal{H}$, is expressed as:

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j+\sum_{j=1}^{\infty}\overline{b_j}\overline{z^j}\,,|b_1|<1 \qquad (z\in\mathbb{U}). \tag{1.1}$$

This function f reduces to l for $k=0$. Jahangiri [10] introduced the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ comprising of functions f such that

$$l(z)=z-\sum_{j=2}^{\infty}|a_j|z^j, \quad (z\in\mathbb{U}). \tag{1.2}$$

and

$$k(z)=\sum_{j=1}^{\infty}|b_j|z^j, \quad (z\in\mathbb{U}). \tag{1.3}$$

A function $f\in\mathcal{T}_{\mathcal{H}}(\alpha)$, if it satisfies the condition:

$$\frac{\partial}{\partial\theta}\left(arg\, f(re^{i\theta})\right)\geq\alpha,\ \ 0\leq\alpha<1. \tag{1.4}$$

Received: March 2018; Accepted: December 2018
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Jahangiri proved that a function f satisfying (1.4) along with

$$\sum_{j=1}^{\infty}\frac{j-\alpha}{1-\alpha}|a_j|+\sum_{j=1}^{\infty}\frac{j+\alpha}{1-\alpha}|b_j|\leq 2, 0\leq\alpha<1, \tag{1.5}$$

where l and k are defined by (1.2) and (1.3) respectively, then f is sense-preserving and convex of order α in $\mathbb{U}$.

Frasin [9] defined the class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma)$ consisting of functions $f: f(z)=l(z)+k(z)$ satisfy the condition:

$$\Re\left\{\alpha z l''(z)+\frac{k(z)}{z}\right\}>1-|\gamma|, \text{ where } \gamma\in\mathbb{C},\ \alpha\geq 0, z\in\mathbb{U},$$

where l and k are of the form (1.2) and (1.3) respectively. He also proved that if $f\in\mathcal{T}_{\mathcal{H}}(\alpha,\gamma)$, then

$$\sum_{j=2}^{\infty}\left[\alpha j(j-1)|a_j|-\frac{1-3\alpha}{j+\alpha}\right]\leq|\gamma|,$$

for $a_1=b_1=1, 0\leq\alpha<\frac{1}{3}$ and $\gamma\in\mathbb{C}$. Let $\mathcal{F}(z)=\mathfrak{L}(z)+\mathcal{K}(z)$, where

$$\mathfrak{L}(z)=z^t-\sum_{j=2}^{\infty}|a_j|z^{j+t-1}(z\in\mathbb{U}) \tag{1.6}$$

and

$$\mathcal{K}(z)=\sum_{j=1}^{\infty}|b_j|z^{j+t-1}(z\in\mathbb{U}). \tag{1.7}$$

Makinde and Afolabi [11] introduced the class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,t)$ consisting of functions $\mathcal{F}(z)=\mathfrak{L}(z)+\mathcal{K}(z)$ such that $\mathfrak{L}$ and $\mathcal{K}$ are of the form (1.6) and (1.7) respectively and satisfying the condition:

$$\Re\left\{\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}\right\}>1-|\gamma|(z\in\mathbb{U}),$$

where $\gamma\in\mathbb{C}$ and $\alpha\geq 0$. In this particular article, Makinde and Afolabi studied various properties of the class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,t)$. In the following, we define a new class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$.

Definition 1.1. *Let* $\mathcal{F}(z)=\mathfrak{L}(z)+\mathcal{K}(z)$ be a harmonic function defined in $\mathbb{U}$ such that $\mathfrak{L}$ and $\mathcal{K}$ have the series representations (1.6) and (1.7) respectively. Then $\mathcal{F}\in\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$, if $\mathfrak{L}$ and $\mathcal{K}$ satisfy the condition:

$$\Re\left(\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}\right)\geq m\left|\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}-1\right|+1-|\gamma|(z\in\mathbb{U}), \tag{1.8}$$

where $\alpha\geq 0, m\geq 0$ *and* $\gamma\in\mathbb{C}$

Functions in the class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$ are related with the uniformly harmonic functions. The image domain of such functions is basically conic depending on the values of m. The condition described above in (1.8) is equivalent to:

$$\Re\left[\left(1+me^{i\theta}\right)\left(\alpha z\mathfrak{L}''(z)+\frac{\mathcal{K}(z)}{z}\right)-me^{i\theta}\right]\geq 1-|\gamma|, \qquad (\pi\leq\theta\leq\pi, z\in\mathbb{U}). \tag{1.9}$$

To avoid repetition of parameters in the Definition1.1, we assume these parameters with the above specific restrictions. If we take $m=0$ in (1.8), we obtain the class $\mathcal{T}_{\mathcal{H}}(\alpha,\gamma,t)$, see [11] with references therein. For other related results, we also refer [1-2, 8, 11-18].

2. PRELIMINARY RESULTS

In this section, we include a useful lemma. This lemma deals with the conditions on the infinite series of coefficients involving in the representation of $\mathcal{F}$.

Lemma 2.1. *Let* $\mathcal{F}(z) = \mathfrak{L}(z) + \mathcal{K}(z)$ *be so that* $\mathfrak{L}$*and* $\mathcal{K}$ *are given by* (1.6)*and* (1.7) *respectively. If* $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha, t)$*, then*

$$\sum_{j=1}^{\infty} \frac{j+t-1-\alpha}{1-\alpha}\left(|a_j| + |b_j|\right) \leq 2,$$

where $a_1 = b_1 = 1, 0 < t \leq 1,$ *and* $0 \leq \alpha < 1$.

3. RESULTS

In the following, we find estimates on the coefficients bounds. These coefficients bounds further lead to the estimates of growth and distortions related to the functions $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, m, t)$.

Theorem 3.1. Let $\mathcal{F}(z) = \mathfrak{L}(z) + \mathcal{K}(z)$, where $\mathfrak{L}$ and $\mathcal{K}$ satisfy (1.6) and (1.7) respectively. If $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, m, t)$, then

$$\sum_{j=2}^{\infty} \left\{ \alpha(j+t-1)(j+t-2)|a_j| - \frac{2-t-3\alpha}{j+t-1+\alpha} \right\} \leq \frac{|\gamma|}{1+m}, \tag{3.1}$$

where $a_1 = b_1 = 1$, $0 < t \leq 1$, $\frac{1}{3} \leq \alpha < \frac{2}{3}$, *and* $\gamma \in \mathbb{C}$.

Proof. Let $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, m, t)$. In view of (1.6), (1.7), (1.9) and for $t_j = j + t - 1$, we have

$$Re\left[(1+me^{i\theta})z^{t-1}\left(\alpha t(t-1) + |b_1| - \sum_{j=2}^{\infty}\{\alpha t_j(t_j - 1)|a_j| - |b_j|\}z^{j-1}\right) - me^{i\theta}\right] \geq 1 - |\gamma|.$$

Choosing z to be real and letting$z \to 1^-$ in the above inequality and simplifying, we obtain

$$(1+m)\left[\alpha t(t-1) + |b_1| - \textstyle\sum_{j=2}^{\infty}\{\alpha(j+t-1)(j+t-2)|a_j| - |b_j|\}\right] \geq 1 - |\gamma| + m.$$

For $0 < \alpha t(t-1) + |b_1| \leq 1$, we write

$$\sum_{j=2}^{\infty}\{\alpha(j+t-1)(j+t-2)|a_j| - |b_j|\} \leq \frac{|\gamma|}{1+m}. \tag{3.2}$$

From Lemma 2.1, we have

$$\sum_{j=1}^{\infty} \frac{j+t-1+\alpha}{1-\alpha}|b_j| \leq \sum_{j=1}^{\infty}\left\{\frac{j+t-1-\alpha}{1-\alpha}|a_j| + \frac{j+t-1+\alpha}{1-\alpha}|b_j|\right\} \leq 2,$$

where $0 \leq \alpha \leq 1, a_1 = 1$. This implies that

$$|b_j| \leq \frac{2-t-3\alpha}{j+t-1+\alpha}, \qquad j \geq 2. \tag{3.3}$$

For $0 < t \leq 1$, $\frac{1}{3} \leq \alpha < \frac{2}{3}, \gamma \in \mathbb{C}$ and on substituting (3.2) into (3.3), we get

$$\sum_{j=2}^{\infty}\{\ \alpha(j+t-1)(j+t-2)|a_j| - \frac{2-t-3\alpha}{j+t-1+\ \alpha}\} \leq \frac{|\gamma|}{1+m}.$$

In the following, we deduce the conditions for the coefficients bounds for functions in the class $\mathcal{T}_{\mathcal{H}}\ (\alpha, \gamma, \mathrm{m}, \mathrm{t})$.

Corollary 3.2. *Let* $\mathcal{F}\ (z) = \mathfrak{L}(z) + \mathcal{K}(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, m, t)$, *where* $\mathfrak{L}$ *and* $\mathcal{K}$ *satisfy* (1.6) *and* (1.7) *respectively. Then*

$$|a_j| \leq \frac{(j+t-1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha(j+t-1)(j+t-2)(j+t-1+\alpha)(1+m)},$$

where $\gamma \in \mathbb{C}$ *and* $j \geq 2$.

The following theorems deal with the growth and distortions related problems of the function $\mathfrak{L}$ involved in the class $\mathcal{T}_{\mathcal{H}}\ (\alpha, \gamma, \mathrm{m}, \mathrm{t})$.

Theorem 3.3. Let $\mathcal{F}(z) = \mathfrak{L}(z) + \mathcal{K}(z) \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, \mathrm{m}, \mathrm{t})$, where $\mathfrak{L}$ and $\mathcal{K}$ satisfy (1.6) and (1.7) respectively. Then

$$\mathrm{r}^{\mathrm{t}} - \theta(\alpha, \gamma, \mathrm{m}, \mathrm{t})|\mathrm{r}|^{\mathrm{t}+1} \leq |\mathfrak{L}(\mathrm{z})| \leq \mathrm{r}^{\mathrm{t}} + \theta(\alpha, \gamma, \mathrm{m}, \mathrm{t})|\mathrm{r}|^{\mathrm{t}+1}, \tag{3.4}$$

where

$$\theta(\alpha, \gamma, m, t) = \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha\ )}{\alpha t(t+1)(1+m)(t+1+\alpha)}.$$

Proof Let $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}\ (\alpha, \gamma, m, t)$. Then from (3.2), we have

$$\alpha t(t+1)\sum_{j=2}^{\infty}|a_j| - \sum_{j=2}^{\infty}|b_j| \leq \frac{|\gamma|}{1+m}. \tag{3.5}$$

From (3.3) and (3.5), we write

$$\sum_{j=2}^{\infty}|a_j| \leq \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(t+1)(1+m)(t+1+\alpha)}. \tag{3.6}$$

From (1.6), we have the following inequality:

$$|\mathfrak{L}(z)| \geq r^t - \sum_{j=2}^{\infty}|a_j||r|^{t+1}\ . \tag{3.7}$$

Combining (3.6) and (3.7), we thus obtain

$$|\mathfrak{L}(z)| \geq r^t - \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(t+1)(1+m)(t+1+\alpha)}|r|^{t+1}. \tag{3.8}$$

Adopting similar procedure, we write

$$|\mathfrak{L}(z)| \leq r^t + \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha\ )}{\alpha t(t+1)(1+m)(t+1+\alpha)}|r|^{t+1}. \tag{3.9}$$

On combining(3.8) and (3.9), we obtain (3.4).

Theorem 3.4. If $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha, \gamma, m, t)$, then for $|z| = r < 1$,we have

$$tr^{t-1} - \theta(\alpha,\gamma,m,t)r^t \le \left|\mathfrak{L}'(z)\right| \le tr^{t-1} + \theta(\alpha,\gamma,m,t)r^t, \qquad (3.10)$$

where

$$\theta(\alpha,\gamma,m,t) = \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(1+m)(t+1+\alpha)}.$$

Proof. For $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$, from (3.2), we write

$$\alpha t \sum_{j=2}^{\infty} (j+t-1)|a_j| \le \frac{|\gamma|}{1+m} + \sum_{j=2}^{\infty} |b_j|. \qquad (3.11)$$

From (3.3) and (3.11), we obtain

$$\sum_{j=2}^{\infty} (j+t-1)|a_j| \le \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(1+m)(t+1+\alpha)} \qquad (3.12)$$

Also for $|z| = r < 1$, we can write

$$|\mathfrak{L}'(z)| \ge tr^{t-1} - \sum_{j=2}^{\infty} (j+t-1)|a_j| r^t. \qquad (3.13)$$

From (3.12) and (3.13), we have the following

$$|\mathfrak{L}'(z)| \ge tr^{t-1} - \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(1+m)(t+1+\alpha)} r^t. \qquad (3.14)$$

Also for $|z| = r < 1$, we obtain that

$$|\mathfrak{L}'(z)| \le tr^{t-1} + \frac{(t+1+\alpha)|\gamma| + (1+m)(2-t-3\alpha)}{\alpha t(1+m)(t+1+\alpha)} r^t. \qquad (3.15)$$

Combining equations(3.14) and (3.15), we have the desired result.
In the theorem, we calculate the radius of starlikeness for $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$.

Theorem 3.5. *Let* $\mathcal{F}(z) = \mathfrak{L}(z) + \mathcal{K}(z) \in \mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$,
where $\mathfrak{L}$ and $\mathcal{K}$ are given by (1.6) and (1.7) respectively. Then $\mathfrak{L}$ is starlike of order ρ $(0 \le \rho < 1)$in $|z| < r_1$, where

$$r_1 = \operatorname*{Inf}_j \left[\frac{(2-\rho-t)(1-\rho)\{\alpha+t-1)|\gamma| + (1+m)(2-t+3\alpha)\}}{\alpha t(1+m)(1-\rho)(t+j-\rho-1)(t+j-1)(t+\alpha+1)}\right]^{\frac{1}{j+t-2}}.$$

Proof. To obtain the desired result, it is enough to show that:

$$\left|\frac{z\mathfrak{L}'(z)}{\mathfrak{L}(z)}-1\right| \leq 1-\rho \text{ for } |z|<r_1$$

From (1.6), we have the following representation:

$$z\mathfrak{L}'(z) = tz^t - \sum_{j=2}^{\infty}(j+t-1)\left|a_j\right|z^{j+t-1}.$$

Thus, on simplifications, we can write

$$\left|\frac{z\mathfrak{L}'(z)}{\mathfrak{L}(z)}-1\right| \leq \frac{(t-1)+\sum_{j=2}^{\infty}(j+t-2)\left|a_j\right||z|^{j+t-2}}{1-\sum_{j=2}^{\infty}\left|a_j\right||z|^{m+t-2}}, \quad (a_1=1).$$

The inequality $\left|\frac{z\mathfrak{L}'(z)}{\mathfrak{L}(z)}-1\right| \leq 1-\rho$ holds only if

$$\frac{t-1}{1-\rho}+\sum_{j=2}^{\infty}\left(\frac{j+t-1-\rho}{1-\rho}\right)\left|a_j\right||z|^{j+t-2} \leq 1. \tag{3.16}$$

The coefficients inequality (3.16) along with (3.12) yield the following bounds

$$|z|^{j+t-2} \leq \frac{(2-\rho-t)(1-\rho)\{\alpha+t-1)|\gamma|+(1+m)(2-t+3\alpha)\}}{\alpha t(1+m)(1-\rho)(t+j-\rho-1)(t+j-1)(t+\alpha+1)},$$

that is,

$$|\mathrm{z}| \leq \left[\frac{(2-\rho-t)(1-\rho)\{\alpha+t-1)|\gamma|+(1+m)(2-t+3\alpha)\}}{\alpha t(1+m)(1-\rho)(t+j-\rho-1)(t+j-1)(t+\alpha+1)}\right]^{\frac{1}{j+t-2}}. \tag{3.17}$$

Hence, from (3.17) we deduce the radius of the starlikeness of the functions $\mathcal{F} \in \mathcal{T}\_\mathcal{H}\ (\alpha,\gamma,m,t)$, that is,

$$r_1 = \inf_j \left[\frac{(2-\rho-t)(1-\rho)\{\alpha+t-1)|\gamma|+(1+m)(2-t+3\alpha)\}}{\alpha t(1+m)(1-\rho)(t+j-\rho-1)(t+j-1)(t+\alpha+1)}\right]^{\frac{1}{j+t-2}}.$$

This completes the proof.

3.5. Remarks

We may also calculate radius of convexity of the functions $\mathcal{F} \in \mathcal{T}_{\mathcal{H}}(\alpha,\gamma,m,t)$.

4. CONCLUSIONS

In this research, we introduced a new class $\mathcal{T}_{\mathcal{H}}\ (\alpha,\gamma,m,t)$of harmonic functions. We obtained the coefficients inequalities, growth and distortion bounds, radius of starlikeness for the analytic part of the harmonic functions involved in this newly defined class. For specific values of parameters involved, our findings may be related to the previously known results.

5. ACKNOWLEDGEMENT

The Authors acknowledge the contributions of Prof. Dr. Habib-ur-Rahman, *FCSP SI* for the promotion of research conducive environment at MUST, Mirpur, AJK.

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