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Research Article

The Reve's Puzzle with Relaxation of The Divine Rule

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Abstract: This paper considers a variant of the Reve's puzzle with $n\ (\geq 1)$ discs which admits of $r\ (\geq 1)$ number of violations of the "divine rule". Denoting by $S_4(n, r)$ the minimum number of moves required to solve the new variant, we give a scheme to find the optimality equation satisfied by $S_4(n, r)$. We then find an explicit form of the optimal value function $S_4(n, r)$.

Keywords: Tower of Hanoi, Divine rule, Sinner's tower, Reve's puzzle.

1. INTRODUCTION

The Tower of Hanoi puzzle with three pegs and *8* discs of varying sizes, invented by the French Number theorist Lucas [1], is well known. An immediate generalization of the Tower of Hanoi problem is the 4-peg variant, which appears as the Reve's puzzle in Dudeney [2]. In general form, the Reve's puzzle is as follows : There are $n\ (\geq 1)$ discs d_1, d_2, ..., d_n of varying sizes, and 4 pegs, S, P_1, P_2 and D. Initially, the discs' rest on the source peg, S, in a tower in increasing order, with the largest disc at the bottom, the second largest disc above it, and so on, with the smallest disc at the top. The problem is to shift the tower from the peg S to the destination peg, D, in minimum number of moves, where each move can transfer only the topmost disc from one peg to another under the "divine rule" that no disc is ever placed on top of a smaller one.

The Tower of Hanoi as well as the 4-peg generalization has seen many variations, some of which have been reviewed by Majumdar [3]. Recently, Chen, Tian and Wang [4] have introduced a new variant of the Tower of Hanoi problem which allows $r\ (\geq 1)$ violations of the "divine rule". In the new variant, the problem is to shift the tower of n discs from the peg S to the peg D in minimum number of moves, where for (at most) r moves, some disc may be placed directly on top of a smaller one. Denoting by $S_3(n, r)$ the minimum number of moves required to solve the new variant, $S_3(n, r)$ is given in the following lemma, due to Chen, Tian and Wang [4].

Lemma 1.1 : For any $n \geq 1,\ r \geq 1,$

$$S_3(n, r) = \begin{cases} 2n-1, & \text{if } 1 \leq n \leq r+2 \\ 4n-2r-5, & \text{if } r+2 \leq n \leq 2r+3 \\ 2^{n-2r}+6r-1, & \text{if } n \geq 2r+3 \end{cases}$$

This paper generalizes the problem of Chen, Tian and Wang [4] to the Reve's puzzle. The problem that we consider here may be stated as follows: Given a tower of $n\ (\geq 1)$ discs on the peg S, the objective is to transfer it to the peg D in minimum number of moves, where the "divine rule" may be violated (at most) r times. Chen, Tian and Wang [4] call their variant as the sinner's tower. Then, the variant we consider may be called the sinner's tower with one Devil peg.

Denoting by $S_4(n, r)$ the minimum number of moves required to solve the Reve's puzzle with $n\ (\geq 1)$ discs and $r\ (\geq 1)$ relaxations of the "divine rule", we find an explicit form of $S_4(n, r)$. This is done in Section 3. In Section 2, we give some background material. In the final Section 4, some observations are made. We also give an open problem, where r number of relaxation of the "divine rule" is allowed.

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2. PRELIMINARY RESULTS

Let $M_4(n)$ denote the minimum number of moves required to solve the Reve's puzzle with $n\ (\geq 1)$ discs. Then, the dynamic programming equation satisfied by $M_4(n)$ is (see, for example, Roth [5], Wood [6], Hinz [7], Chu and Johnson baugh [8], and Majumdar [9, 10]):

$$M_4(n) = \min_{1 \leq k \leq n-1} \{ 2M_4(K)+2^{n-k} -1 \},\ n \geq 4, \quad (2.1a)$$

With

$$M_4(0) = 0;\ M_4(n) = 2n - 1 \text{ for all } 1 \leq n \leq 3. \quad (2.1b)$$

Lemma 2.1: Exactly one of the following two relationships hold:

(1) $M_4(n + 2) - M_4(n + 1) = M_4(n + 1) - M_4(n),$

(2) $M_4(n + 2) - M_4(n + 1) = 2\{M_4(n + 1) - M_4(n)\}.$

The following two corollaries are the consequences of Lemma 2.1.

Corollary 2.1:

$M_4(n + 1) - M_4(n) = 2$ *if and only if* $n = 1, 2.$

Proof: It is easy to show the "if" part of the lemma. Now, since for any $n \geq 3$,

$$M_4(n + 1) - M_4(n) \geq M_4(4) - M_4(3) = 4 > 2 = M_4(3) - M_4(2),$$

The result follows, by virtue of Lemma 2.1.

Corollary 2.2:

For $n \geq 2r + 3,\ M_4(n - r) - M_4(n - 2r) \geq 4r.$

Proof: Since

$$M_4(n-r)-M_4(n-2r)=\sum_{i=0}^{r-1} \left[M_4(n-r-i)-M_4(n-r-i-1)\right]$$

$\geq r\,[M_4(n - 2r + 1) - M_4(n - 2r)],$

we see that, for $n \geq 2r + 3$,

$$M_4(n - r) - M(n - 2r) \geq r[M_4(4) - M_4(3)] = 4r.$$

The solution of the optimality equation (2.1) is given below for future reference (for a proof, the reader is referred to Majumdar [9, 10]).

Theorem 2.1: Let

$$\frac{s(s + 1)}{2} < n < \frac{(s + 1)(s + 2)}{2}$$

for some $s \in \{1, 2, ...\}$.

Then,

$$(1)\ M_4\left(\frac{s(s + 1)}{2}\right) = 2^s(s-1)+1,$$

attained at the unique points $k = \frac{s(s-1)}{2}$.

$$(2)\ M_4(n) = 2^s\left\{ n - \frac{s(s - 1)}{2} - 1 \right\} + 1,$$

attained at the two points $k = n - s - 1,\ n - s.$

Lemma 2.2: Let the function $F(k)$ be defined as follows:

$$F(k) = M_4(k) - 2k,\ k \geq 0.$$

Then,

(1) $F(k)$ is strictly increasing in $k \geq 2,$

(2) $F(k)$ attains its minimum (with the minimum value – 1) at the points $k = 1, 2, 3.$

Proof: Since

$$F(k + 1) - F(k) = [M_4(k + 1) - M_4(k)] - 2,$$

part (1) follows immediately by virtue of Corollary 2.1.

Then, part (2) is an easy exercise, and is left for the reader.

Let us consider the following optimization problem:

$$\text{Min } \{2M_4(k) + 6\ell + 2m + 1\} \quad (2.2)$$

such that

$$k + 2\ell + m = n - 1$$
$$\ell + m - 1 = r$$
$$0 \leq k \leq n - 1,\ \ell \geq 0,\ m \geq 0$$

Lemma 2.3: The optimization problem (2.2) is equivalent to the

$$\text{Min } 2\{M_4(k) - 2k\} + 4n - 2r - 5 \quad (2.3)$$

such that

$$k = n - 2r + m - 3$$
$$\ell + m = r + 1$$
$$0 \le k \le n - 1,\ \ell \ge 0,\ m \ge 0$$

with the minimum value

$$\begin{cases} 4n-2r-7, & \text{if } r+4 \le n \le 2r+6 \\ 2M_4(n-2r-3)+6r+7, & \text{if } n \ge 2r+7 \end{cases} \quad (2.4)$$

Proof : From the two equality constraints in (2.2), we get after eliminating ℓ,

$$k = n - 2r + m - 3. \quad (2.5)$$

Using the constraint conditions $\ell + m = r + 1$ and (2.5), we may re-write the objective function in (2.2) as follows:

$$2M_4(k) + 6\ell + 2m + 1 = 2M_4(k) + 6(r - m + 1) + 2m + 1$$

$$= 2M_4(k) + 6r - 4(k - n + 2r + 3) + 7$$

$$= 2M_4(k) - 4k + 4n - 2r - 5.$$

Now, if $r + 4 \le n \le 2r + 6$, then from (2.5), $m - r + 1 \le k \le m + 3$, and we may choose $m \in \{0, 1, ..., r + 1\}$ such that $k \in \{1, 2, 3\}$. Then, for any such k, $F(k)$ of Lemma 2.2 attains the minimum value – 1, and hence, the objective function in (2.3) has the minimum value $4n - 2r - 7$. On the other hand, if $n \ge 2r + 7$ (so that $k \ge m + 4$), part (2) of Lemma 2.3 asserts that the objective function in (2.3) is strictly increasing in k, and hence, it attains its minimum at $k = n - 2r - 3$. Then, after simplifying, we get (2.4).

Thus, the lemma is established.
It may be mentioned here that, when $r+4 \le n \le 2r+6$, by properly choosing m, we may have $k \in \{1, 2, 3\}$. For example, in the extreme case $n = r + 4$, choosing $m = r$ in (2.5), we get $k = 1$. Another extreme case is $n = 2r + 6$, where $m = 0$ gives $k = 3$.

3. THE PROBLEM & ITS SOLUTION

Formally, the problem that we consider is as follows: There are four pegs, S, P_1, P_2 and D. Initially, there is a tower of n (≥ 1) discs (of varying sizes) on the source peg S, in small-on-large ordering. The objective is to move this tower to the destination peg D, using the auxiliary pegs P_1 and P_2, in minimum number of moves, where each move shifts the topmost disc from one peg to another, and for (at most) r (≥ 1) moves, some disc may be placed directly on top of a smaller one.

Let $S_4(n, r)$ be the minimum number of moves required to solve the above problem. The following theorem gives an explicit form of $S_4(n, r)$.

Theorem 3.1: For $n \ge 1$, $r \ge 1$,

$$S_4(n, r) = \begin{cases} 2n-1, & \text{if } 1 \le n \le r+3 \\ 4n-2r-7, & \text{if } r+4 \le n \le 2r+6 \\ M_4(n-2r)+6r, & \text{if } n \ge 2r+7 \end{cases}$$

Proof : The proof is trivial if $1 \le n \le 3$.
So, let $4 \le n \le r + 3$. In this case, the transfer of the tower from the peg S to the peg D may be affected as follows :

✔ Scheme 1

1. Move the topmost $n - 3$ ($\le r$) discs from the peg S to the peg P_1, one by one, in an "inverted tower" (thereby violating the "divine rule" at most $r - 1$ times).

2. Shift the next two largest discs on the peg S to the peg P_2 in an "inverted tower", which violates the "divine rule" once.

3. Transfer the largest disc d_n from the peg S to the peg D.

4. Move the discs on the peg P_2 to the peg D.

5. Finally, shift the discs on the peg P_1, one by one, to the peg D, to complete the tower on the peg D.

The total number of violations of the "divine rule" is (at most) r, and the total number of moves involved is

$$2\{(n-3)+2\}+1=2n-1.$$

Next, let $r+4 \le n \le 2r+6$. In this case, we follow the scheme below :

✔ Scheme 2

1. Move the top most k (≥ 0) discs, $d_1, d_2, \ldots, d_k$, from the source peg S to some auxiliary peg, say, P_1, in a tower in $M_4(k)$ moves.
2. Consider the next 2ℓ $(\ell \ge 1)$ discs on the peg S. With these 2ℓ discs, form ℓ pairs of discs (d_i, d_{i+1}). For each pair (d_i, d_{i+1}), d_i is first moved to the peg D, next d_{i+1} is shifted to the peg P_2, and then d_i is moved again (from the peg D) to the peg P_2. Note that, in this step, the first pair does not violate the "divine rule", but each of the next $\ell - 1$ pairs violate the "divine rule" once. This step requires 3ℓ moves, and the "divine rule" is violated $\ell - 1$ times (so that ℓ satisfies the condition that $1 \le \ell \le r+1$).
3. Move the next m (≥ 0) largest discs (from the peg S) to the peg P_2, one by one, in an "inverted tower", in m moves, violating the "divine rule" m times.
4. Transfer the largest disc d_n (from the peg S) to the peg D.
5. The m discs in the "inverted tower" on P_2 are shifted, one by one, to D.
6. For each of the ℓ pairs of discs (d_i, d_{i+1}) on the peg P_2, di is moved to the peg S, next d_{i+1} is shifted to the peg D, and then di is moved again (from S) to D.
7. Finally, move the k discs from the peg P_1 to the peg D, in a tower.

The total number of moves involved in the above 7 steps is:

$$2\{M_4(k)+3\ell+m\}+1=2M_4(k)+6\ell+2m+1,$$

and the total number of violations of the "divine rule" is $\ell + m - 1$, where the numbers k $(0 \le k \le n-1)$, ℓ $(1 \le \ell \le r+1)$, and m $(0 \le m \le r)$ are to be determined so as to minimize the total number of moves. Thus, the above scheme leads to the optimization problem (2.2), or, equivalently, (2.3). Now, for $r+4 \le n \le 2r+6$, the result follows from Lemma 2.3.

Finally, let $n \ge 2r+7$. We consider the following scheme to transfer the tower from the peg S to the peg D.

✔ Scheme 3

1. Move the topmost k (≥ 1) discs, $d_1, d_2, \ldots, d_k$, from the peg S to some auxiliary peg P_1, say, using the four pegs available, in (minimum) $M_4(k)$ moves.
2. Shift the remaining $n-k$ discs on the peg S to the peg D, using the three pegs available, in (minimum) $S_3(n-k, r)$ moves.
3. Finally, transfer the tower of k discs from the peg P_1 to the peg D, again in (minimum) $M_4(k)$ moves, to complete the tower on the destination peg D.

The total number of moves involved is, using Lemma 1.1,

$$2M_4(k)+S_3(n-k, r)=2M_4(k)+2^{n-k-2r}+6r-1,$$

and k is to be determined such that the total number of moves is minimum. Thus, in this scheme, the minimum number of moves required is:

$$\min_{1 \le k \le n-2r} [2M_4(k)+2^{n-2r-k}+6r-1]=M_4(n-2r)+6r,$$

where we have used (2.1a).

Letting

$$n=2r+7+t,\ t \ge 0,$$

the (minimum) number of moves under Scheme 2 is, by virtue of Lemma 2.3, $2M_4(t+4)+6r+7$, while, the (minimum) number of moves is $M_4(t+7)+6r$ under Scheme 3. Since $M_4(6)$ in (2.1) is attained at the (unique) point $k=3$ and $M_4(10)$ is attained at the (unique) point $k=6$, it follows that:

$$M_4(t+7) < 2M_4(t+4)+7 \text{ for all } t \ge 3.$$

It is an easy exercise to verify, using Theorem 2.1, that

$$M_4(t+7)=2M_4(t+4)+7 \text{ for all } 0 \le t \le 2$$

.

All these complete the proof of the theorem.

Remark 3.1. In addition to Scheme 2 and Scheme 3 above, there is another one to shift the tower from the peg S to the peg D, namely, the following one:

1. Move the topmost $k\ (\geq 0)$ discs from the peg S to the peg P_1, say, in (minimum) $M_4(k)$ moves.
2. Shift the next r largest discs $d_{k+1}, d_{k+2}, \ldots, d_{k+r}$ from the peg S to the peg P_1, in an "inverted tower" (violating the "divine rule" r times).
3. Transfer the tower of $n - k - r$ discs from the peg S to the peg D, (using the three available pegs) in (minimum) $2^{n-k-r} - 1$ moves.
4. Move the discs $d_{k+r}, d_{k+r-1}, \ldots, d_{k+1}$, in this order, one by one, from P_1 to D.
5. Finally, shift the tower (of k discs) on the peg P_1 to the peg D.

The minimum number of moves required under this scheme is

$$\min_{1 \leq k \leq n-r} [2\{M_4(k) + r\} + 2^{n-r-k} - 1] = M_4(n-r)+2r.$$

However, note that, by Corollary 2.2, $M_4(n - 2r) + 6r \leq M_4(n - r) + 2r$ for all $n \geq 2r + 3$, so that this scheme is worse than Scheme 3.

It may be mentioned here that, by symmetry, Step 2 and Step 3 in Scheme 2 may be interchanged; in this case, $0 \leq \ell \leq r$, and $1 \leq m \leq r + 1$, and Step 5 and Step 6 are to be interchanged as well.

4. CONCLUSION

From the proof of Theorem 3.1, we observe that, when $n = r + 3$ (so that $k = m - r \geq 0$), we have the "saturated case" of "inverted tower" in the sense that all the topmost $n - 1$ discs are placed in "inverted tower" on the auxiliary peg (S) just before the largest disc is moved (from the peg S) to the peg D. Again, when $n = 2r + 6$ (so that $k = m + 3 \geq 3$), we have the "saturated case" in Step 2 in Scheme 2 in the sense that all the r number of violations of the "divine rule" is used up in this step. This shows that, for $n \geq 2r + 7$, for each increase in n, we have to increase the number of discs in Step 1 accordingly.

From Theorem 3.1, we observe further that, for $n \geq 2r + 7$, the function $S_4(n, r)$ involves $M_4(n - 2r)$, and so for any $r \geq 1$ fixed, we may appeal to Theorem 2.1 to find the expression of $S_4(n, r)$. It is indeed interesting to find that the new variant has a closed-form solution, given in Theorem 3.1, and further that the optimal value function can be expressed in terms of the optimal value function of the original Reve's puzzle.

To see how the relaxation of the "divine rule" affects the original optimal value function, we consider the case when $r = 1$. From Theorem 3.1, we see that

$$S_4(n,1) = \begin{cases} 2n-1, & \text{if } 1 \leq n \leq 4 \\ 4n-9, & \text{if } 5 \leq n \leq 8 \end{cases}$$

and for $n \geq 8$,

$$S_4(n, 1) = M_4(n-2) + 6.$$

Let

$$n - 2 = \frac{s(s+1)}{2} + R \text{ for some integer } s \geq 3,$$

where $0 \leq R \leq s$. Then, by Theorem 2.1,

$$M_4\left(\frac{s(s+1)}{2} + R\right) = 2^s\left\{\frac{s(s+1)}{2} + R - \frac{s(s-1)}{2} - 1\right\} + 1$$

$$= 2^s(s + R - 1) + 1.$$

Therefore,

$$S_4(n,1) = S_4\left(\frac{s(s+1)}{2} + R + 2,\ 1\right) = 2^s(s + R - 1) + 7. \quad (4.1)$$

Since

$$M_4(n) = 2^s(s + R + 1) + 1, \quad (4.2)$$

from (4.1) and (4.2), we see that, the relaxation of the "divine rule" once, the number of moves decreases approximately by an additive factor of 2^{s+1}. It may be mentioned here that, in some cases, there are multiple optimal strategies. For example, when $n = 2r + 4$, an alternative scheme is the following :

1. Move the disc d_1 from the source peg S to some auxiliary peg, say, P_1.

2. Consider the next $2(r + 1)$ discs on the peg S. With these discs, form $r + 1$ pairs of discs (d_i, d_{i+1}). For each pair (d_i, d_{i+1}), d_i is first moved to the peg D, next d_{i+1} is shifted to the peg P_2, and then d_i is moved again (from the peg D) to the peg P_2. Note that, in this step, the first pair does not violate the "divine rule", but each of the next r pairs violates the "divine rule" once. This step requires $3(r + 1)$ moves,

Table 1. Values of $S_4(n, r)$ *for,* $n = 1(1)10$, $r = 0(1)6$

r / n	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	3
3	5	5	5	5	5	5	5
4	9	7	7	7	7	7	7
5	13	11	9	9	9	9	9
6	17	15	13	11	11	11	11
7	25	19	17	15	13	13	13
8	33	23	21	19	17	15	15
9	41	31	25	23	21	19	17
10	49	39	29	27	25	23	21

and the "divine rule" is violated $r - 2$ times.

3. Transfer the largest disc d_n (from the peg S) to the peg D.
4. For each of the $(r + 1)$ pairs of discs (di, d_{i+1}) on the peg P_2, d_i is moved to the peg S, next d_{i+1} is shifted to the peg D, and then d_i is moved again (from S) to D.
5. Finally, move the disc from the peg P_1 to the peg D.

The scheme requires:

$$2[1 + 3(r + 1)] + 1 = 6r + 9$$

number of moves.

It is an interesting problem to look for all the alternative optimal schemes. It may be noted here that, for $n \geq 2r + 10$, Scheme *3* is the only optimal policy. Chen, Tian and Wang [4] have posed the Tower of Hanoi problem with an evildoer disc. Another problem of interest is the following generalization:

Reve's Puzzle with r Evildoers: In the Reve's puzzle, any r of the $n \geq 1$ discs may be an evildoer, where an evildoer disc can be placed directly on top of a smaller disc any number of times.

Denoting by $E(n, r)$ the minimum number of moves required to solve the above problem, it is found that

$$E(n, 1) = S_4(n, 1) \text{ for } 1 \leq n \leq 17,$$

but $E(18, 1) = 155$, if the disc D_{16} is taken as the evildoer. It remains open to find an expression of $E(n, r)$. For small values of n and r, the values of $S_4(n, r)$ can be calculated easily. Table 1 gives the values of $S_4(n, r)$ for $n = 1(1)10$, $r = 0(1)6$. For $r \geq 7$, the number of moves is $2n - 1$, $1 \leq n \leq 10$.

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