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Research Article

A Solution of the Navier-Stokes Problem for an Incompressible Fluid

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Abstract: It is known that the methods of integral transformations in the theory of partial differential equations made it possible to find solutions to many problems and clarify the physical meaning of some basic laws and phenomena in fluid mechanics. In this regard, in the present work, we study the Navier-Stokes system, which describe the flow of a viscous incompressible fluid. Moreover, on the basis of the developed method, the original problem is transformed to the system of Volterra and Volterra-Abel integral equations of the second kind, and taking into account the theory of these systems, the existence and uniqueness of the solution of the non-stationary Navier-Stokes problem in the special space, which was introduced in the paper, are proved. The solution was obtained for velocity and pressure in an analytical form, in addition, the found pressure distribution law, which is described by a Poisson type equation and plays a fundamental role in the theory of Navier-Stokes systems in constructing analytic smooth (conditionally smooth) solutions.

Keywords: Navier-Stokes Equation, Partial Differential Equations (PDE), Incompressible Fluid, Inhomogeneous Linear Equations, Solution Uniqueness.

1. INTRODUCTION

The difficulty in solving the 3D Navier-Stokes equations is due to their nonlinearity and the need to find the velocity and pressure depending on any values of the viscosity parameter [1], but despite this there are numerous works in this direction with certain limitations. For example, in works [2, 3 and 4], showed that the Navier–Stokes equations in three space dimensions always have a weak solution with suitable growth properties. Scheffer [5] applied ideas from geometric measure theory to prove a partial regularity theorem for suitable weak solutions of the Navier–Stokes equations. The partial regularity theorem of concerns a parabolic analogue of the Hausdorff dimension of the singular set of a suitable weak solution of Navier–Stokes. In this paper, we are not trying to consider the extensive references on the Navier-Stokes system, since there are fundamental works in this area (see, for example, Landau-Lifshitz [6], Ladyzhenskaya [7], Prantdl [8], Schlichting [9] and others). Therefore, we restrict attention here to incompressible fluids filling all of R^3. The Navier–Stokes equations are then given by:

$$\frac{\partial v}{\partial t}+(v\nabla)v=f-\frac{1}{\rho}\operatorname{grad}P+\mu\Delta v, \qquad (1.1)$$

$$\operatorname{div} v=0, \forall(x,t)\in\bar{D}_0=R^3\times[0,T_0], \qquad (1.2)$$

with initial conditions

$$v\big|_{t=0}=\varphi(x)\lambda,\ \forall x\in R^3, \qquad (1.3)$$

where $\varphi(x)$ is known scalar function, $R^3 \ni \lambda$ is given vector with positive constant components: $0<\lambda_i,(i=\overline{1,3}), R^3 \ni f(x,t)$ is external applied force (e.g. gravity), $0<\mu$ is kinematic viscosity, ρ is density, Δ is Laplace operator, ∇ is Hamilton operator. These equations are to be solved for an unknown velocity vector $v\in R^3$ and pressure $P(x,t)$, and equation (1.2) just says that the fluid is incompressible. Moreover, the purpose of this paper is

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to establish the existence and uniqueness of a solution to the Navier-Stokes system for an incompressible fluid in space $G_3^1(D_0)$ by the norm:

$$\begin{cases} \|\nu\|_{G_3^1(D_0)} = \sum_{i=1}^{3}\|\nu_i\|_{G^1(D_0)} = \sum_{i=1}^{3}\{\sum_{0\le|k|\le 2}\|D^k\nu_i\|_{C(\bar{D}_0)} + \\ +\|\nu_{it}\|_1\}, \\ \|\nu_{it}\|_1 = \sup_{R^3}\int_0^{T_0}|\nu_{it}(x,t)|dt, (i=\overline{1,3}), \end{cases}$$

where $k=(k_1,k_2,k_3)$ is the multi-index,

$$\begin{cases} \nu=(\nu_1,\nu_2,\nu_3),\ k=0: D^0\nu_i\equiv\nu_i, \\ k\ne 0: D^k\nu_i = \dfrac{\partial^{|k|}\nu_i}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}},\ (i=\overline{1,3}), \\ |k|=\sum_{j=1}^{2}k_j,\ (k_j=0,1,2;\ j=1,2). \end{cases} \qquad (1.4)$$

In this regard, we note that in our early works, for example, in [10], we proposed method for constructing smooth (conditionally smooth) solutions of nD Navier-Stokes equations in $G_n^1(D_0=R^n\times(0,T_0))$ with the condition:

$$\nu|_{t=0}=\psi(x),\forall x\in R^n,\ (\psi\in R^n,\ n\ge 3). \qquad (1.5)$$

Since it takes place

$$\begin{aligned} &\int_{R^n}\psi(\tau)ds+\int_0^{T_0}\int_{R^n}f(\tau,s)d\tau ds=\lambda, \\ &(f\in R^n,\ \tau\in R^n), \end{aligned} \qquad (1.6)$$

here $R^n \ni \lambda$ is a known vector with positive constant components: $0<\lambda_i,(i=\overline{1,n})$, and then applying the transformation:

$$\nu=\theta\lambda+(\exp(-\frac{t}{\mu\delta_0}))J(x,t), \qquad (1.7)$$

where $\theta(x,t)$ is the new unknown scalar function, and δ_0 is the introduced constant, which ensures the application of the Banach principle and the Picard's method [11] for the system of integral equations of Volterra-Abel type of the second kind, into which the original problem is transformed, $R^n \ni J(x,t)$ is the given vector:

$$\begin{cases} J=\dfrac{1}{2^n(\sqrt{\mu\pi t})^n}\int_{R^n}\psi(\tau)\exp(-\dfrac{|x-\tau|^2}{4\mu t})d\tau, \\ |x-\tau|=\sqrt{\sum_{i=1}^{n}(x_i-\tau_i)^2},\ (x,\tau\in R^n), \\ J|_{t=0}=\psi(x),\ \forall x\in R^n, \\ 0<\mu<1;\ \ 0<\delta_0=const<1, \end{cases}$$

moreover, (1.7) is consistent at the initial time with condition (1.5) and with the incompressibility condition (1.2), (when $\nu\in R^n, n\ge 3)$. Therefore, since it takes place:

$$\begin{cases} \theta|_{t=0}=0,\ \ \forall x\in R^n;\ \mathrm{div}\,\nu=0,\ \mathrm{div}\psi=0: \\ \mathrm{div}J=\dfrac{1}{\sqrt{\pi^n}}\int_{R^n}\exp(-|\xi|^2)\mathrm{div}\psi(x+ \\ +2\xi\sqrt{\mu t})d\xi=0,\ (\tau=x+2\xi\sqrt{\mu t}\in R^n), \\ 0=\mathrm{div}\,\nu=\mathrm{div}(\theta\lambda)+(\exp(-\dfrac{t}{\mu\delta_0}))\times \\ \times\mathrm{div}J=\mathrm{div}(\theta\lambda)=\sum_{i=1}^{n}\theta_{x_i}\lambda_i, \\ (\theta\lambda\nabla)\theta\lambda=\lambda_i\theta(\sum_{j=1}^{n}\lambda_j\theta_{x_j})=0,(i=\overline{1,n}). \end{cases} \qquad (1.8)$$

Then, taking into account (1.1), (1.7) and (1.8) it follows, $(\nu \in R^n, n \geq 3):$

$$(\nu\nabla)\nu = (\theta\lambda\nabla)\theta\lambda + (\exp(-\frac{t}{\mu\delta_0}))\times \\ \times[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \\ +(\exp(-\frac{2t}{\mu\delta_0}))(J\nabla)J = (\exp(-\frac{t}{\mu\delta_0}))\times \\ \times[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + (\exp(-\frac{2t}{\mu\delta_0}))(J\nabla)J, \tag{1.9}$$

and this means that under condition (1.2), the inertial terms of equation (1.1), taking into account (1.7), are linearized with respect to the newly introduced function $\theta(x,t)$ and its derivatives with respect to $x \in R^n$, and the nonlinearity goes over to the known vector of the function $J(x,t)$ and partial derivatives with respect to $x \in R^n$. Therefore,

$$\frac{\partial\theta}{\partial t}\lambda + (\exp(-\frac{t}{\mu\delta_0}))[(\theta\lambda\nabla)J + \\ +(J\nabla)\theta\lambda] = \frac{1}{\mu\delta_0}(\exp(-\frac{t}{\mu\delta_0}))J - \\ -(\exp(-\frac{2t}{\mu\delta_0}))(J\nabla)J + f - \frac{1}{\rho}\operatorname{grad} P + \\ +(\mu\Delta\theta)\lambda, \tag{1.10}$$

since

$$\begin{cases} J = \dfrac{1}{2^n(\sqrt{\mu\pi t})^n}\displaystyle\int_{R^n}\psi(\tau)\exp(-\frac{|x-\tau|^2}{4\mu t})d\tau, \\ G(x,\tau,t) \equiv \dfrac{1}{2^n(\sqrt{\mu\pi t})^n}\exp(-\dfrac{|x-\tau|^2}{4\mu t}),(t>0), \\ L[G] \equiv \dfrac{\partial G}{\partial t} - \mu\Delta G = 0, \end{cases}$$

in addition, the Poisson equation for pressure is derived in the form:

$$\begin{cases} \dfrac{1}{\rho}\Delta P = -\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\nu_{ix_j}\nu_{jx_i} = -\{F_0 + \\ +(\exp(-\dfrac{t}{\mu\delta_0}))[\displaystyle\sum_{i=1}^{n}(\sum_{j=1}^{n}\lambda_j J_{ix_j})\theta_{x_i} + \\ +\displaystyle\sum_{i=1}^{n}(\sum_{j=1}^{n}J_{jx_i}\theta_{x_j})\lambda_i]\}, \\ \operatorname{div} f = 0, \\ F_0 \equiv (\exp(-\dfrac{2t}{\mu\delta_0}))\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}J_{ix_j}J_{jx_i}, \end{cases} \tag{1.11}$$

and it is obtained on the basis of (1.7) by applying the operation div to equation (1.1), (or (1.10)). As a result, equation (1.1) on the basis of (1.7), (1.10) and (1.11) is transformed to a system of Volterra and Volterra-Abel equations of second kind, where the solvability of the original problem in $G_n^1(D_0)$ follows from the solvability of this system. Then in the same paper, by the Sobolev theorem [11], suitable solutions of the Navier-Stokes problem were constructed in space $W_{n;\lambda}^2(D_* = R^n \times (0,\infty))$, that is, in this case the Navier-Stokes system admits a unique weak global solution in time in $W_{n;\lambda}^2(D_*)$. Theoretically, an equation of this kind with low viscosity is of scientific interest among mechanics and mathematicians.

2. FLUID WITH SMALL VISCOSITY

There are various mathematical transformations in the theory of the partial differential equations which simplify investigated problems and allow us to find the solutions in certain spaces ([6, 7] and [11]). In this regard, let the components of a velocity vector $v(x,t)$ and $f_i,(i=1,2,3)$ be the components of a given, externally applied force, satisfy the conditions(1.3) and,

$$\begin{cases} f \in R^3,\ \varphi \in R,\ \lambda \in R^3:\ \ \mathrm{div} f = 0, \\ \mathrm{div}(\lambda\varphi) = 0, \\ \left|D^k \varphi\right| \le \beta_0 = const,\ \forall x \in R^3, \\ \left|D^k f_i\right| \le \beta_1 = const,\ \forall (x,t) \in \bar{D}_0,\ (i = \overline{1,3}). \end{cases} \tag{2.1}$$

In this case, we seek the solution of the Navier-Stokes problem in the form:

$$\mathrm{v} = \theta\lambda + \mu J(x,t), \tag{2.2}$$

where $R^3 \in J$ is known vector-valued function of the form:

$$\begin{cases} J = \dfrac{1}{2^3\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} \dfrac{1}{\sqrt{(\mu(t-s))^3}} f(\tau,s) \times \\ \times \exp\left(-\dfrac{r^2}{4\mu(t-s)}\right) d\tau ds, (x,\tau \in R^3), \\ 0 < \mu < 1;\ r = |x-\tau| = \sqrt{\sum\limits_{i=1}^{3} (x_i - \tau_i)^2}, \end{cases} \tag{*}$$

i.e., the vector $J(x,t)$ satisfies the differential heat equation with the homogeneous Cauchy condition, i.e.:

$$\begin{cases} J_t = f + \mu\Delta J, \\ J\big|_{t=0} = 0, \forall x \in R^3, \end{cases} \tag{2.3}$$

and $\theta(x,t)$ is a new unknown scalar function with the condition:

$$\theta\big|_{t=0} = \varphi(x),\ \forall x \in R^3. \tag{2.4}$$

Lemma 1. In case of (2.2), when conditions (1.2), (2.1) are satisfied, the inertial terms of equation (1.1), taking into account (2.2), are linearized with respect to the introduced function $\theta(x,t)$ and its derivatives.

Proof. In fact, under conditions (1.2) and (2.1), it follows from (2.2):

$$\begin{cases} \mathrm{div} f = 0:\ \mathrm{div} J = \dfrac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} \mathrm{div} f(x + \\ +2\xi\sqrt{\mu(t-s)}, s) \exp(-|\xi|^2) d\xi ds = 0, \\ \tau = x + 2\xi\sqrt{\mu(t-s)} \in R^3;\ \mathrm{div}\, \mathrm{v} = 0: \\ 0 = \mathrm{div}\, \mathrm{v} = \mathrm{div}\theta\lambda + \mu \mathrm{div} J = \sum\limits_{i=1}^{3} \lambda_i \theta_{x_i}. \end{cases} \tag{2.5}$$

And since

$$(\theta\lambda\nabla)\theta\lambda = \lambda_i\theta\left(\sum_{j=1}^{3} \lambda_j \theta_{x_j}\right) = 0, (i = \overline{1,3}), \tag{2.6}$$

then, on the basis of (2.2), (2.5) and (2.6), the inertial terms of equation (1.1) are equivalently converted to the form:

$$\begin{aligned} (\mathrm{v}\nabla)\mathrm{v} &= (\theta\lambda\nabla)\theta\lambda + \mu[(\theta\lambda\nabla)J + \\ &+ (J\nabla)\theta\lambda] + \mu^2 (J\nabla)J = \mu[(\theta\lambda\nabla)J + \\ &+ (J\nabla)\theta\lambda] + \mu^2 (J\nabla)J. \end{aligned} \tag{2.7}$$

The indicated transformation is natural, since the incompressibility of the original problem is characterized with condition (1.2). Which was required to show.

Further, substituting (2.2) into equation (1.1), we obtain a linear inhomogeneous differential equation of the type of heat conduction with variable coefficients:

$$\frac{\partial\theta}{\partial t}\lambda + \mu[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2 (J\nabla)J =$$

$$=(1-\mu)f-\frac{1}{\rho}\text{grad}\,P+(\mu\Delta\theta)\lambda. \qquad (2.8)$$

From where the equation for pressure is derived:

$$\begin{cases}\frac{1}{\rho}\Delta P=-\sum_{i=1}^{3}\sum_{j=1}^{3}\nu_{ix_j}\nu_{jx_i}=-\{F_0+\\ +\mu[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i}+\sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i]\}, \qquad (2.9)\\ F_0\equiv\mu^2\sum_{i=1}^{3}\sum_{j=1}^{3}J_{ix_j}J_{jx_i}.\end{cases}$$

We are taking into account the operation div with respect to (2.8), (that's tantamount to applying the operation div with respect to equation (1.1), since (1.1) is equivalently converted to the form (2.8) based on (2.2)).

In this case,

$$\begin{cases}\lambda_1^{-1}\sum_{j=1}^{3}\lambda_j J_{1x_j}\equiv\lambda_2^{-1}\sum_{j=1}^{3}\lambda_j J_{2x_j}\equiv\lambda_3^{-1}\sum_{j=1}^{3}\lambda_j J_{3x_j},\\ \lambda_1^{-1}\{\frac{1}{\rho}P_{x_1}-f_1(1-\mu)+\mu^2\sum_{j=1}^{3}J_jJ_{1x_j}\}\equiv\\ \equiv\lambda_2^{-1}\{\frac{1}{\rho}P_{x_2}-f_2(1-\mu)+\mu^2\sum_{j=1}^{3}J_jJ_{2x_j}\}\equiv \qquad (2.10)\\ \equiv\lambda_3^{-1}\{\frac{1}{\rho}P_{x_3}-f_3(1-\mu)+\mu^2\sum_{j=1}^{3}J_jJ_{3x_j}\},\end{cases}$$

(2.10) is the condition of unequivocal compatibility for case (2.8), since θ is a scalar function. Therefore, since there are (2.1), (2.2), (2.7) and system (2.8), then the pressure is determined by rule (2.9), since when applying the operation *div* to equation (2.8), the equalities holds:

$$\begin{cases}\rho^{-1}\text{div}(\text{grad}\,P)=\rho^{-1}\Delta P,\\ \text{div}f=0,\ \text{div}(\theta_t\lambda)=0,\\ \text{div}(\mu\Delta\theta)\lambda=0,\ \text{div}(\Delta J)=0,\ (\text{div}J=0),\\ \text{div}\{\mu[(\theta\lambda\nabla)J+(J\nabla)\theta\lambda]+\mu^2(J\nabla)J\}=F_0+\\ +\mu(\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i}+\sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i).\end{cases}$$

Here, formula (2.9) modifies the Landau – Lifshitz formula (see [6]: (15.11)) and is an equation of Poisson type. Then it follows from (2.9),

$$\begin{cases}P(x,t)=\int_{R^3}\frac{1}{r}\rho\Omega(\tau,t)d\tau,(x,\tau\in R^3),\\ r=|x-\tau|;\ \Omega(x,t)\equiv\frac{1}{4\pi}\{F_0+ \qquad (2.11)\\ +\mu[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i}+\sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i]\},\end{cases}$$

at that

$$\frac{\partial}{\partial x}P=\int_{R^3}\rho\Omega(\tau,t)\frac{\partial\frac{1}{r}}{\partial x}d\tau=$$

$$=\int_{R^3}\rho\Omega(\tau,t)\frac{\tau-x}{r^3}d\tau,(\tau-x\in R^3), \qquad (2.12)$$

where (2.11) is called the Newtonian potential [11]. On the other hand, a solution to the Poisson equation (2.9) tending to zero at infinity will be unique if the function $\theta_{x_i},(i=1,2,3)$ is unique, since the function $\Omega(x,t)$ contains these functions.

To prove the above, we note that the obtained pressure distribution law allows us to express the velocity in integral form when $\nu\in R^3$. In fact, substituting (2.12) into equation (2.8) with allowance for (2.10), we obtain an inhomogeneous linear integro-differential heat conduction equation with the Cauchy condition:

$$\begin{cases}\theta_t = \Phi + \mu B[\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}] + \mu\Delta\theta, \\ \theta\big|_{t=0} = \varphi(x), \forall x \in R^3,\end{cases} \tag{2.13}$$

here the known functions contained in (2.13) are introduced on the basis of the notation:

$$\begin{cases}\Phi \equiv \Phi_1 + \Phi_2, \\ \Phi_1 \equiv d_0^{-1}\sum_{i=1}^{3}(1-\mu)f_i(x,t), \\ d_0 = \sum_{i=1}^{3}\lambda_i > 0; \ \ \Omega(x,t), \quad \text{(see. (2.11))}: \\ \Phi_2(x,t) \equiv d_0^{-1}[-\mu^2\sum_{i=1}^{3}(\sum_{j=1}^{3}J_jJ_{ix_j}) - \\ -\frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\tilde{\xi}_i}{r_1^3}F_0(x+\tilde{\xi};t)d\tilde{\xi}], \\ r_1 = \sqrt{(\tilde{\xi}_1^2+\tilde{\xi}_2^2+\tilde{\xi}_3^2)^3}; \ \ h = x+\tilde{\xi} \in R^3, \\ B[\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}] \equiv -\{d_0^{-1}\theta(.)\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{is_j}) + \\ +\sum_{j=1}^{3}\theta_{x_j}(.)I_j(.) + d_0^{-1}(\frac{1}{4\pi}\int_{R^3}\sum_{k=1}^{3}\frac{\tilde{\xi}_k}{r_1^3}\times \\ \times[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ih_j}(x+\tilde{\xi},t))\theta_{h_i}(x+\tilde{\xi},t) + \\ +\sum_{i=1}^{3}(\sum_{j=1}^{3}I_{jh_i}(x+\tilde{\xi},t)\theta_{h_j}(x+\tilde{\xi},t))\lambda_i]d\tilde{\xi})\}.\end{cases} \tag{2.14}$$

It is known that problem (2.9) with sufficiently smooth initial data is solvable [11, 12] in $G^1(D_0)$, i.e. the solution of the problem under study is reduced to the determination of function θ from the equation:

$$\theta = \Upsilon + \frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\times \\ \times\mu(B[\theta,\theta_{\tau_1},\theta_{\tau_2},\theta_{\tau_3}])(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3}, \tag{2.15}$$

where Υ is a known function:

$$\begin{cases}\Upsilon \equiv \sum_{i=0}^{2}\Upsilon_i; \ \ \Phi \equiv \Phi_1 + \Phi_2, \quad \text{(see. (2.14))}: \\ \Upsilon_0 = \frac{1}{\sqrt{\pi^3}}\int_{R^3}\exp(-|\xi|^2)\varphi(x+2\xi\sqrt{\mu t})d\xi, \\ \Upsilon_1 = \frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-|\xi|^2))\Phi_1(x+2\xi\times \\ \times\sqrt{\mu(t-s)},s)d\xi ds; \\ \Upsilon_2 = \frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\times \\ \times\Phi_2(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3}, \\ \Phi_2(x,t): \\ \Upsilon_{2x_i} = \frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\times \\ \times\Phi_2(\tau,s)\frac{\tau_i - x_i}{2\mu(t-s)}\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3} = \\ = -\frac{d_0^{-1}\mu\sqrt{\mu}}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-|\xi|^2))\frac{\xi_i}{\sqrt{t-s}}\times \\ \times[\sum_{m=1}^{3}(\sum_{j=1}^{3}J_j(x+2\xi\sqrt{\mu(t-s)},s)J_{mx_j}(x+ \\ +2\xi\sqrt{\mu(t-s)},s)) + \frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\tilde{\xi}_i}{r_1^3}\times \\ \times\sum_{m=1}^{3}\sum_{k=1}^{3}J_{mx_k}(x+2\xi\sqrt{\mu(t-s)}+\tilde{\xi};s)\times \\ \times J_{kx_m}(x+2\xi\sqrt{\mu(t-s)}+\tilde{\xi};s)d\tilde{\xi}]\times \\ \times d\xi ds \equiv \Phi_{3,i}(x,t), (i=\overline{1,3}), \\ \Upsilon_{2x_i^2} = \frac{\partial}{\partial x_i}\Phi_{3,i}(x,t), (i=\overline{1,3}), \\ \Upsilon_{0t} = \frac{1}{\sqrt{\pi^3}}\int_{R^3}(\exp(-|\xi|^2))\sum_{k=1}^{3}\frac{\xi_k}{\sqrt{t}}\sqrt{\mu}\times \\ \times\varphi_{l_k}(x+2\xi\sqrt{\mu t})d\xi, \ (t>0),\end{cases} \tag{2.16}$$

and it is easy to see, since Φ is a smooth function of spatial coordinates, then the function $\Upsilon(x,t)$ admits restrictions:

$$\begin{cases} \left|D^k\varphi\right| \le \beta_0,\ \left|D^k f_i\right| \le \beta_1,\ (i=\overline{1,3}), \\ \left|D^k\Phi\right| \le \beta_2;\ \left|D^k\Upsilon\right| \le \beta_3,\ (x,t)\in \bar{D}_0, \\ \sup \frac{1}{\sqrt{\pi^3}} \int\limits_{R^3} (\exp(-|\xi|^2)) \sum\limits_{k=1}^{3} |\xi_k| d\xi \le \beta_{4,1}, \\ \sup\limits_{[0,T_0]} \frac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \sum\limits_{k=1}^{3} \frac{1}{\sqrt{t-s}} \times \\ \times |\xi_k| d\xi ds \le \beta_{4,2}, \\ \beta_4 = max(\beta_{4,1};\beta_{4,2}), \\ \|\Upsilon_{0t}\|_1 = \sup\limits_{R^3} \int\limits_0^{T_0} |\Upsilon_{0t}(x,t)| dt \le 2\beta_0\beta_4\sqrt{\mu T_0}, \\ \Upsilon_t = \Phi + \frac{1}{\sqrt{\pi^3}} \int\limits_{R^3} (\exp(-|\xi|^2)) \sum\limits_{k=1}^{3} \frac{\xi_k}{\sqrt{t}} \times \\ \times \sqrt{\mu}\varphi_{l_k}(x+2\xi\sqrt{\mu t}) d\xi + \\ + \frac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \sum\limits_{k=1}^{3} \frac{\xi_k}{\sqrt{t-s}} \sqrt{\mu} \times \\ \times \Phi_{l_k}(x+2\xi\sqrt{\mu(t-s)},s) d\xi ds,\ (t>0), \\ \|\Upsilon_t\|_1 = \sup\limits_{R^3} \int\limits_0^{T_0} |\Upsilon_t(x,t)| dt \le 2\beta_0\beta_4\sqrt{\mu T_0} + \\ +T_0(\beta_2 T_0 + \sqrt{\mu}\beta_2\beta_4) \le \beta_5, \\ \|\Upsilon\|_{G^1(D_0)} = \sum\limits_{0\le|k|\le 2} \|D^k\Upsilon\|_{C(\bar{D}_0)} + \|\Upsilon_t\|_1 \le \beta_6, \\ 0<\beta_i = const,\ i=\overline{0,6} = const, \end{cases} \tag{2.17}$$

i.e., $\Upsilon \in G^1(D_0)$.

Further, since equation (2.15) contains

$\theta, \theta_{x_i}, (i=1,2,3)$,

then, taking into account

$$\theta_{x_i} = W_i,\ (i=1,3), \tag{2.18}$$

equation (2.15) is supplemented to a system,

$$\begin{cases} \theta = \Upsilon + \frac{1}{2^3\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-\frac{r^2}{4\mu(t-s)})) \times \\ \times \mu(B[\theta,W_1,W_2,W_3])(\tau,s) \frac{d\tau ds}{(\sqrt{\mu(t-s)})^3} = \\ = \Upsilon + \frac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \mu \times \\ \times (B[\theta,W_1,W_2,W_3])(x+2\xi\sqrt{\mu(t-s)},s) \times \\ \times d\xi ds \equiv \Gamma_0[\theta,W_1,W_2,W_3], \\ W_i = \Upsilon_{x_i} + \frac{1}{2^3\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-\frac{r^2}{4\mu(t-s)})) \times \\ \times \frac{-(x_i-\tau_i)}{2\mu(t-s)} \mu(B[\theta,W_1,W_2,W_3])(\tau,s) \times \\ \times \frac{d\tau ds}{(\sqrt{\mu(t-s)})^3} = \Upsilon_{x_i} + \frac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} \frac{\sqrt{\mu}}{\sqrt{t-s}} \times \\ \times \xi_i(\exp(-|\xi|^2))(B[\theta,W_1,W_2,W_3])(x+ \\ +2\xi\sqrt{\mu(t-s)},s) d\xi ds \equiv \\ \equiv \Gamma_i[\theta,W_1,W_2,W_3],\ (i=\overline{1,3}), \\ \tau = x+2\xi\sqrt{\mu(t-s)} \in R^3. \end{cases} \tag{2.19}$$

The system (2.19) consists of the Volterra and Volterra-Abel integral equations of the second kind. Therefore, constructing approximate solutions to such systems, there are various methods that are well known in the mathematics references, for example [11 and 12] and others.

Since, under the conditions of problem (1.1) - (1.3), the introduced Voltaire type operators $\Gamma_i, (i=0,1,2,3)$ of system (2.19) contain small viscositie μ and $\sqrt{\mu}$, then the proof of solvability and the construction of the approximate solution can be realized on the basis of the Banach principle and the

Picard's method [11], when conditions are allowed:

$$\begin{cases} \Gamma_0 : \ d_0 = \mu L_{\Gamma_0} \le \sqrt{\mu} L_{\Gamma_0} < \dfrac{d}{4}, \\ \Gamma_i : d_i = \sqrt{\mu} L_{\Gamma_i} < \dfrac{d}{4}, \ (d<1; i=\overline{1,3}), \\ \sum_{j=0}^{3} \sqrt{\mu} L_{\Gamma_j} = k_0\sqrt{\mu} < d, \\ k_0 = \sum_{j=0}^{3} L_{\Gamma_j}, \ (0<\mu<\min(1, d^2 k_0^{-2})), \\ \|\Gamma_0[\theta_0, W_{1,0}, W_{2,0}, W_{3,0}] - \theta_0\|_C \le r(1-d), \\ \|\Gamma_j[\theta_0, W_{1,0}, W_{2,0}, W_{3,0}] - W_{j,0}\|_C \le \\ \le r(1-d), (j=\overline{1,3}), \\ S_r(\theta_0) = \{\theta : |\theta-\theta_0| \le r, \forall (x,t) \in \bar{D}_0\}, \\ S_r(W_{j,0}) = \{W_i : |W_i - W_{j,0}| \le r, \\ \forall (x,t) \in \bar{D}_0\}, (j=\overline{1,3}), \end{cases} \tag{2.20}$$

where $\theta_0, W_{j,0}$ is initial estimates, L_{Γ_j} is the Lipschitz coefficient of the operator $\Gamma_j, (j=0,1,2,3)$, and here with

$$\begin{cases} E = \|\theta\|_C + \|W_1\|_C + \|W_2\|_C + \|W_3\|_C, \\ E \le (1-d)^{-1} M_1 = M_2, (M_1 = 4\beta_3). \end{cases} \tag{2.21}$$

Uniqueness of functions $\theta, W_i, (i=1,2,3)$ follows from the solvability of system (2.19), moreover, taking into account expressions $B[\theta, W_1, W_2, W_3]$ of formula (2.14) from (2.19) we have:

$$\begin{cases} W_{ix_i} = \Upsilon_{x_i^2} - \sqrt{\mu}\dfrac{1}{\sqrt{\pi^3}} \int_0^t \int_{R^3} (\exp(-|\xi|^2)) \times \\ \times \dfrac{\xi_i}{\sqrt{(t-s)}} \{ d_0^{-1}[W_i(x+2\xi\sqrt{\mu(t-s)}, s) \times \\ \times \sum_{j=1}^{3} (\sum_{m=1}^{3} \lambda_m J_{jh_m}(x+2\xi\sqrt{\mu(t-s)}, s)) + \\ +\theta(x+2\xi\sqrt{\mu(t-s)}, s) \dfrac{\partial}{\partial x_i} (\sum_{j=1}^{3}\sum_{m=1}^{3} \lambda_m \times \\ \times J_{jh_m}(x+2\xi\sqrt{\mu(t-s)}, s))] + \sum_{m=1}^{3} [W_{mx_i}(x+ \\ +2\xi\sqrt{\mu(t-s)}, s) J_m(x+2\xi\sqrt{\mu(t-s)}, s) + \\ +W_m(x+2\xi\sqrt{\mu(t-s)}, s) J_{mx_i}(x+2\xi \times \\ \times \sqrt{\mu(t-s)}, s)] + d_0^{-1}(\dfrac{1}{4\pi}\int_{R^3} \sum_{k=1}^{3} \dfrac{\tilde{\xi}_k}{r_1^3} \times \\ \times [(\dfrac{\partial}{\partial x_i} \sum_{j=1}^{3}\sum_{m=1}^{3} \lambda_m J_{jh_m}(x+2\xi\sqrt{\mu(t-s)} + \\ +\tilde{\xi}, s)) W_j(x+2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s) + \\ +\sum_{j=1}^{3}\sum_{m=1}^{3} \lambda_m J_{jh_m}(x+2\xi\sqrt{\mu(t-s)} + \\ +\tilde{\xi}, s)) W_{ix_i}(x+2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s) + \\ +(\dfrac{\partial}{\partial x_i} \sum_{j=1}^{3}\sum_{m=1}^{3} \lambda_j J_{mh_j}(x+2\xi\sqrt{\mu(t-s)} + \\ +\tilde{\xi}, s)) W_m(x+2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s) + \\ +\sum_{j=1}^{3}\sum_{m=1}^{3} \lambda_j J_{mh_j}(x+2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s) \times \\ \times W_{mx_i}(x+2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s)] d\tilde{\xi})\} d\xi ds, \\ x \in R^3, \ \xi \in R^3, \ \tilde{\xi} \in R^3; \ x+2\xi \times \\ \times \sqrt{\mu(t-s)} + \tilde{\xi} \in R^3. \end{cases} \tag{2.22}$$

Consequently, we have an estimate of the form:

$$\begin{cases} \tilde{E} \le (1-d)^{-1}[3\beta_3 + 2k_0\sqrt{\mu}E] \le \\ \le (1-d)^{-1}[3\beta_3 + 2k_0\sqrt{\mu}M_2] \le M_3, \\ \tilde{E} = \sum_{j=1}^{3} \|W_{jx_j}\|_{C(\bar{D}_0)}. \end{cases} \tag{2.23}$$

Therefore, differentiating the first equation (2.19) with respect to $x_i, (i=1,2,3)$, we obtain:

$$\begin{cases} \theta = \Upsilon + \dfrac{1}{2^3\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-\dfrac{r^2}{4\mu(t-s)})) \times \\ \times \mu(B[\theta, W_1, W_2, W_3])(\tau, s) \times \\ \times \dfrac{1}{(\sqrt{\mu(t-s)})^3} d\tau ds, \\ \theta_{x_i} = \Upsilon_{x_i} + \dfrac{1}{2^3\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-\dfrac{r^2}{4\mu(t-s)})) \times \\ \times \dfrac{-(x_i - \tau_i)}{2\mu(t-s)} \mu(B[\theta, W_1, W_2, W_3])(\tau, s) \times \\ \times \dfrac{1}{(\sqrt{\mu(t-s)})^3} d\tau ds = \Upsilon_{x_i} + \\ + \dfrac{1}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \dfrac{\xi_i}{\sqrt{t-s}} \times \\ \times \sqrt{\mu}(B[\theta, W_1, W_2, W_3])(x + 2\xi \times \\ \times \sqrt{\mu(t-s)}, s) d\xi ds, \\ x, \xi \in R^3; \ \tau = x + 2\xi\sqrt{\mu(t-s)} \in R^3. \end{cases} \qquad (2.24)$$

Since the right side of (2.24) is a known function, then, taking into account (2.21), it follows from (2.24):

$$\begin{cases} \|\theta\|_C \le \beta_3 + d_0 M_2 = M_4, \\ \|\theta_{x_i}\|_C \le \beta_3 + d_i M_2 \le M_5, \\ M_5 = \beta_3 + \tilde{d} M_2; \ \max d_i = \tilde{d}, (i = \overline{1,3}). \end{cases} \qquad (2.25)$$

And this means that the functions $\theta, \theta_{x_i}, (i=1,2,3)$ are uniquely determined from (2.24), since the functions $\theta, W_i, (i=1,2,3)$ are unique. Therefore, pressure is the only one by formula (2.11). Which was required to show.

On the other hand, since (2.22) with estimate (2.23) take place, then the function $\theta_{x_i^2}, (i=1,2,3)$ is defined in the form: $\theta_{x_i^2} = W_{ix_i}$, here in estimate takes place:

$$\|\theta_{x_i^2}\|_{C(\bar{D}_0)} \le M_6, \ (i = \overline{1,3}). \qquad (2.26)$$

Taking into account that the function θ has second-order continuous partial derivatives with respect to spatial coordinates, and estimates (2.25), (2.26), we have:

$$\sum_{0 \le |k| \le 2} \|D^k \theta\|_{C(\bar{D}_0)} \le M_7. \qquad (2.27)$$

Then, taking the time derivative from the first equation of (2.19), we obtain:

$$\begin{cases} \theta_t = \Upsilon_t + \mu B[\theta, \theta_{x_1}, \theta_{x_2}, \theta_{x_3}] + \dfrac{1}{\sqrt{\pi^3}} \mu \times \\ \times \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \sum\limits_{j=1}^3 \dfrac{\partial}{\partial t} \{(B[\theta, \theta_{h_1}, \theta_{h_2}, \theta_{h_3}])(x + \\ + 2\xi\sqrt{\mu(t-s)}, s)\} d\xi ds = \Upsilon_t + \\ + \mu B[\theta, \theta_{x_1}, \theta_{x_2}, \theta_{x_3}] - \dfrac{\mu\sqrt{\mu}}{\sqrt{\pi^3}} \int\limits_0^t \int\limits_{R^3} (\exp(-|\xi|^2)) \times \\ \times \sum\limits_{i=1}^3 \dfrac{\xi_i}{\sqrt{(t-s)}} \{d_0^{-1}[W_{h_i}(x + 2\xi\sqrt{\mu(t-s)}, s) \times \\ \times \sum\limits_{j=1}^3 (\sum\limits_{m=1}^3 \lambda_m J_{jh_m}(x + 2\xi\sqrt{\mu(t-s)}, s)) + \\ + \theta(x + 2\xi\sqrt{\mu(t-s)}, s) \sum\limits_{j=1}^3 \sum\limits_{m=1}^3 \lambda_m J_{jh_m h_i}(x + \\ + 2\xi\sqrt{\mu(t-s)}, s)] + \sum\limits_{m=1}^3 [W_{mh_i}(x + 2\xi \times \\ \times \sqrt{\mu(t-s)}, s) J_m(x + 2\xi\sqrt{\mu(t-s)}, s) + \\ + W_m(x + 2\xi\sqrt{\mu(t-s)}, s) J_{mh_i}(x + 2\xi \times \\ \times \sqrt{\mu(t-s)}, s)] + d_0^{-1}(\dfrac{1}{4\pi} \int\limits_{R^3} \sum\limits_{k=1}^3 \dfrac{1}{r_1^3} \tilde{\xi}_k [\sum\limits_{j=1}^3 \sum\limits_{m=1}^3 \lambda_m \times \\ \times J_{jh_m \tilde{h}_i}(x + 2\xi\sqrt{\mu(t-s)} + \tilde{\xi}, s) W_j(x + \end{cases}$$

$$\begin{cases} +2\xi\sqrt{\mu(t-s)}+\tilde{\xi},s)+ \\ +\sum_{j=1}^{3}\sum_{m=1}^{3}\lambda_m J_{jh_m}(x+2\xi\sqrt{\mu(t-s)}+ \\ +\tilde{\xi},s)W_{i\tilde{h}_i}(x+2\xi\sqrt{\mu(t-s)}+\tilde{\xi},s)+ \\ +\sum_{j=1}^{3}\sum_{m=1}^{3}\lambda_j J_{mh_j\tilde{h}_i}(x+2\xi\sqrt{\mu(t-s)}+ \\ +\tilde{\xi},s)W_m(x+2\xi\sqrt{\mu(t-s)}+\tilde{\xi},s)+ \\ +\sum_{j=1}^{3}\sum_{m=1}^{3}\lambda_j J_{mh_j}(x+2\xi\sqrt{\mu(t-s)}+ \\ +\tilde{\xi},s)W_{m\tilde{h}_i}(x+2\xi\sqrt{\mu(t-s)}+ \\ +\tilde{\xi},s)]d\tilde{\xi}\,)\}d\xi ds,\ (t>0), \\ x,\xi,\tilde{\xi}\in R^3,\ h=x+2\xi\sqrt{\mu(t-s)}\in R^3, \\ \tilde{h}=x+2\xi\sqrt{\mu(t-s)}+\tilde{\xi}\in R^3, \end{cases} \tag{2.28}$$

moreover, from the estimate (2.28), on the basis of (2.16), (2.23) and (2.25) it follows:

$$\begin{cases} \|\theta_t\|_1=\sup_{R^3}\int_0^{T_0}|\theta_t(x,t)|dt\le M_8, \\ \|\Upsilon_t\|_1\le\beta_5,\ \text{see. (2.17).} \end{cases} \tag{2.29}$$

Therefore, taking into account (2.27) and (2.29), we obtain:

$$\|\theta\|_{G^1(D_0)}=\sum_{0\le|k|\le2}\|D^k\theta\|_{C(\bar{D}_0)}+\|\theta_t\|_1\le M_0, \quad 0<M_0=M_7+M_8. \tag{2.30}$$

Theorem 1. Let the Navier-Stokes system (1.1) is defined on the $\bar{D}_0=R^3\times[0,T_0]$ and with prescribed initial data (1.2), (1.3), and conditions (2.1), (2.10), (2.17) and (2.20). Then there exists a unique solution of the system (2.19) in $G^1(D_0)$. Moreover, taking into account (2.2), there exists solution to problem (1.1), (1.2) and (1.3) in $G_3^1(D_0)$.

Remark 1. Let $\nu\in R^3$ is the velocity vector satisfies condition (1.2) and

$$\nu|_{t=0}=0,\ \forall x\in R^3, \tag{2.31}$$

at that $f_i(x,t)$ is the component of a given external force f admits the conditions:

$$\begin{cases} f=(f_1,f_2,f_3),\ \ |D^k f_i|\le\beta_1=const,\ (i=\overline{1,3}), \\ x\in R^3,\forall(x,t)\in\bar{D}_0;\ \operatorname{div}f=0, \\ \int_0^{T_0}\int_{R^3} f(x,t)dxdt=\lambda, \\ \lambda\in R^3:\ 0<\lambda_i=const,\ (i=\overline{1,3}), \end{cases}$$

and this means that $R^3\ni\lambda$ is a vector with constant components: $0<\lambda_i,(i=1,2,3)$, therefore, it becomes possible to use method (2.2) of the previous paragraph, $\theta(x,t)$ is a new unknown scalar function with the condition:

$$\theta|_{t=0}=0,\ \forall x\in R^3. \tag{2.32}$$

Then, having held a similar discussion, as in the case (2.15), we have all the conditions of theorem 1, here:

$\Upsilon_0(x,t)\equiv0$, since $\varphi(x)\equiv0, x\in R^3$, (see (2.16) and (2.32)).

Remark 2. Let $\nu\in R^3$ is the velocity vector satisfies conditions (1.2), (1.3), and $f_i(x,t)$ is the component of a given external force f admits the conditions:

$$\begin{cases} f\in R^3:\ \operatorname{div}f=0;\ \sum_{j=1}^{3}\lambda_j f_{ix_j}=0,\ (i=\overline{1,3}), \\ R^3\ni\lambda:\ 0<\lambda_i=const. \end{cases} \tag{2.33}$$

Then, based on (2.2) and (*), with respect to a new unknown scalar function θ, it follows the condition (2.4). Therefore, taking into account (1.2), (2.5), (2.6) and (2.33), the inertial terms of equation (1.1) are equivalently converted to the form:

$$(v\nabla)v = (\theta\lambda\nabla)\theta\lambda + \mu[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \\ +\mu^2(J\nabla)J = \mu(J\nabla)\theta\lambda + \mu^2(J\nabla)J, \qquad (2.34)$$

where (2.34) differs from (2.7). Constraints on external force of the form (2.33) make it possible to simplify the Navier-Stokes problem and transform it into a system of Volterian type integral equations of the second kind. In fact, on the basis of (2.2) and (2.34), from (1.1) follows the equation:

$$\frac{\partial\theta}{\partial t}\lambda + \mu(J\nabla)\theta\lambda + \mu^2(J\nabla)J = (1-\mu)f - \\ -\frac{1}{\rho}\operatorname{grad} P + (\mu\Delta\theta)\lambda, \qquad (2.35)$$

and the equation for pressure is derived:

$$\begin{cases} \frac{1}{\rho}\Delta P = -4\pi F_0(x,t),\ (x\in R^3), \\ F_0 \equiv \mu^2\sum_{i=1}^{3}\sum_{k=1}^{3} J_{ix_k}J_{kx_i}. \end{cases} \qquad (2.36)$$

Since θ is a scalar function, then the condition:

$$\lambda_1^{-1}\{\frac{1}{\rho}P_{x_1} - f_1(1-\mu) + \mu^2\sum_{j=1}^{3}J_jJ_{1x_j}\} \equiv \\ \equiv \lambda_2^{-1}\{\frac{1}{\rho}P_{x_2} - f_2(1-\mu) + \mu^2\sum_{j=1}^{3}J_jJ_{2x_j}\} \equiv \\ \equiv \lambda_3^{-1}\{\frac{1}{\rho}P_{x_3} - f_3(1-\mu) + \mu^2\sum_{j=1}^{3}J_jJ_{3x_j}\}, \qquad (2.37)$$

is a univocal compatibility condition for (2.35). On the other hand, we note that it follows from (2.36):

$$P = \frac{1}{4\pi}\rho\int_{R^3}\frac{1}{r}F_0(\tau,t)d\tau, (r = |x-\tau|), \qquad (2.38)$$

here (2.38) tends to zero at infinity, and there are second-order partial continuous derivatives, and for the first-order partial derivatives it takes place:

$$\frac{\partial}{\partial x}P = \frac{1}{4\pi}\rho\int_{R^3}F_0(\tau,t)\frac{\partial\frac{1}{r}}{\partial x}d\tau. \qquad (2.39)$$

In addition, in this case, the pressure becomes known, since the right side (2.38) is a known function, which is the difference between the results of this section and the previous one. Therefore, excluding pressure from (2.35), we obtain a linear differential equation with variable coefficients and with the Cauchy condition of the form:

$$\begin{cases} \theta_t = \Phi - \mu\sum_{j=1}^{3}\theta_{x_j}J_j + \mu\Delta\theta, \\ \theta|_{t=0} = \varphi(x), \forall x \in R^3, \end{cases} \qquad (2.40)$$

where

$$\begin{cases} \Phi \equiv \Phi_1 + \Phi_2;\quad \Phi_1 \equiv d_0^{-1}\sum_{j=1}^{3}(1-\mu)f_j(x,t), \\ \Phi_2 \equiv d_0^{-1}[-\mu^2\sum_{j=1}^{3}(\sum_{m=1}^{3}J_mJ_{jx_m}) - \\ -\frac{1}{4\pi}\int_{R^3}F_0(x+\xi,t)\sum_{k=1}^{3}\frac{1}{|\xi|^3}\xi_k d\xi], \\ d_0 = \sum_{i=1}^{3}\lambda_i > 0;\ x\in R^3,\ \xi\in R^3,\ \tau = x+\xi \in R^3. \end{cases}$$

Further, the solution of the problem under study is reduced to the determination of function from the equation:

$$\theta = \Upsilon - \frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\times$$
$$\times\mu\sum_{j=1}^{3}\theta_{\tau_j}(\tau,s)J_j(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3}, \quad (2.41)$$

where Υ is a known function (see. (2.16)).

If, when studying problems (1.1), (1.2) and (1.3) with condition (2.33), we partially abandon the requirements for the smoothness of the solution in the domain, then the question arises of which functions can be called solutions of the equation (1.1). For this purpose, let the functions φ,Φ be continuous and $div\lambda\varphi = 0$, then equation (2.41) is not reduced to the form (2.19). Therefore, since equation (2.41) contains $\theta_{x_i},(i=1,2,3)$, therefore, equation (2.41) can be integrated in parts, taking into account that the integrands tend to zero at infinity. So (2.41) is reduced to the Volterra-Abel integral equation of the second kind with respect to the function θ, i.e.:

$$\theta = \Upsilon + \sqrt{\mu}H\theta \equiv (\Gamma\theta)(x,t), \quad (2.42)$$

where

$$\begin{cases} -\frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\mu\sum_{j=1}^{3}\theta_{\tau_j}(\tau,s)\times \\ \times J_j(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3} = \mu\frac{1}{2^3\sqrt{\pi^3}}\times \\ \times\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))\{\sum_{j=1}^{3}[\frac{x_j-\tau_j}{2\mu(t-s)}\times \\ \times J_j(\tau,s)+J_{j\tau_j}(\tau,s)]\theta(\tau,s)\}\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3} = \\ = \left|\sum_{j=1}^{3}J_{j\tau_j}=0\right| \equiv \sqrt{\mu}(H\theta)(x,t), \end{cases}$$

$$\begin{cases} H\theta \equiv \frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-|\xi|^2))\{\sum_{j=1}^{3}\frac{\xi_j}{\sqrt{\eta}}J_j(x+ \\ +2\xi\sqrt{\mu\eta},t-\eta)\theta(x+2\xi\sqrt{\mu\eta},t-\eta)\}d\xi d\eta, \\ R^3 \ni h = x+2\xi\sqrt{\mu\eta} \in R^3, \\ t-s=\eta. \end{cases}$$

Under the conditions

$$\begin{cases} \Gamma: L_\Gamma = \sqrt{\mu}L_H < 1, \\ 0<\mu<min(1,\frac{1}{(L_H)^2}), \\ S_r(0) = \{\theta: |\theta| \le r, \forall(x,t)\in\bar{D}_0\}, \\ \Gamma: S_r(0)\to S_r(0), \\ \|\Upsilon\|_{C(\bar{D}_0)} \le \gamma = const, \end{cases} \quad (2.43)$$

equation (2.42) is solvable in $C(\bar{D}_0)$ and this solution is constructed by the Picard's Method:

$$\theta_{n+1} = \Gamma\theta_n, (n=0,1,...).$$

Definition 1. A generalized solution of equation (2.40) in a domain D_0 is any continuous solution to equation (2.42) in $\bar{D}_0$, and since (2.42) has a unique solution, then solution (2.40) is unique.

Definition 2. Under the conditions of Definition 1, a generalized solution of the original problem is a function v constructed by the rule (2.2).

Remark 3. In the case when the functions φ,Φ are continuous, and $\mathrm{div}(\lambda\varphi)=0$ the result is valid, if we understand the partial derivatives in the sense of S. L. Sobolev [11]. This fact is also one of the significant advantages of the applied method.

4. CONCLUSIONS

The main idea of this paper is that the Navier-Stokes equations (1.1) is reduced to Cauchy problem for inhomogeneous linear equations with the variable coefficients of the heat conduction type, based on the transformation (2.2), taking into account conditions (1.2) and (2.1). The indicated conditions are an important factor for the linearization of equation (1.1), since (2.7) holds when formula (2.2) introduced, i.e. the inertial terms in the Navier-Stokes equations with respect to the new unknown function θ and its derivatives $\theta_{x_i}, (i = 1,2,3)$ are linearized. Further, taking into account (2.2), we also obtain Poisson type equations for pressure of the form (2.9), which modifies the Lipschitz-Landau formula. Therefore, with the exclusion of pressure from equation (2.8), the linear parabolic problem (2.13) follows, which is reduced to the system of Volterra and Volterra-Abel integral equations of the second kind (2.19). Note that the proposed method to solve this problem is applied for the first time. The solutions of the transformed equations are regular with respect to the viscosity coefficient μ, and they simplify the analysis of the original problem in space $G_3^1(D_0)$.

On the other hand, since the Navier-Stokes equations with arbitrary initial conditions were studied in paper [10], (see (1.5)), and in this paper these equations are studied with conditions (1.3) and (2.31). And this means that, in fact, from the obtained results of these works it follows that the Navier-Stokes equations for an incompressible fluid with Cauchy conditions are transformed to well-known mathematical problems. Note that in the future, space $G_3^1(D_0)$ can be used for the Navier-Stokes problem in a bounded domain, when D_0 is bounded.

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