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Research Article

A Recurrence Relation Related to the Reve's Puzzle

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Abstract: In a recent paper, Majumdar [1] studied, to some extent, the generalized recurrence relation, introduced by Matsuura [2] :

$$MT(n,\beta) = \min_{1 \le s \le n} \left\{\beta\, MT(n-s,\beta) + 2^s - 1\right\},$$

where $n \ge 1$ and $\beta \ge 2$ are integers. It may be mentioned here that, $\beta = 2$ corresponds to the Reve's puzzle, introduced by Dudeney [3]. In this paper, we study more closely the properties of the function $MT(n, \beta)$, and give a closed-form expression of it when $\beta = 2^i$ (for any integer $i \ge 2$).

Keywords: Reve's puzzle, Recurrence relation, Local-value relationships.

1. INTRODUCTION

Matsuura [2] posed the recurrence relation below:

$$MT(n,\beta) = \min_{0 \le k \le n-1} \left\{\beta\, MT(k,\ \beta) + 2^{n-k} - 1\right\}; \quad n \ge 1, \tag{1.1}$$

$$MT(0,\beta) = 0, \tag{1.2}$$

where $\beta \ge 2$ is an integer.

Some of the properties satisfied by $MT(n, \beta)$ has been studied by Majumdar [1]. In what follows, let, for $n \ge 1$ and $\beta \ge 2$ fixed:

$$FT(n, k, \beta) = \beta\, MT(n, k, \beta) + 2^{n-k} - 1\ ,\ 0 \le k \le n-1.$$

Note that:

$$MT(n,\beta) = \min_{0 \le k \le n-1} \left\{FT(n,\ k,\ \beta)\right\}.$$

The main results found by Majumdar [1] are reproduced below for reference later.

***Lemma 1.1*:** For any $\beta \ge 2$ fixed, let $FT(n, k, \beta)$ and $FT(n + 1, k, \beta)$ be minimized at the points $k = k_1$ and $k = k_2$ respectively. Then, $k_1 \le k_2 \le k_1 + 1$.

The inequality in Lemma 1.1 above needs some explanation. When $\beta = 2^i$ (for some integer $i \ge 2$), as Lemma 2.4 shows, there are instances when $MT(n, \beta)$ is attained at a unique value of k, while in other cases, $MT(n, \beta)$ is attained at exactly two (consecutive) values of k. Thus, in the latter case, k_1 may be interpreted as the minimum of the two values of k at which $MT(n, \beta)$ is attained and then k_2 is the minimum of the two values (if such a situation arises) at which $MT(n, \beta)$ is attained. Alternatively, k_1 may be taken as the maximum of the two values at which $MT(n, \beta)$ is attained, and then k_2 is the maximum of the two values where $MT(n, \beta)$ is attained.

The following lemma shows that, for $\beta\ (\ge 3)$ fixed, $MT(n, \beta)$ is convex with respect to n (in the sense of the left-hand side inequality).

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***Lemma 1.2*:** For any $\beta \ge 3$ fixed, for all $n \ge 1$,

$$MT(n+1, \beta) - MT(n, \beta) \le MT(n+2, \beta) - MT(n+1, \beta) \le 2[MT(n+1, \beta) - MT(n, \beta)].$$

***Lemma 1.3*:** If $\beta = 2^i$ (for some integer $i \ge 1$), $MT(n, \beta)$ is attained either at a unique k, or else at two (consecutive) values.

***Lemma 1.4*:** For any $\beta \ge 3$ and $n \ge 1$ fixed, $FT(n, k, \beta)$ is convex in k in the sense that:

$$FT(n, k+2, \beta) - FT(n, k+1, \beta) \ge FT(n, k+1, \beta) - FT(n, k, \beta)$$

for all $0 \le k \le n-3$.

This paper gives an explicit expression of $MT(n, \beta)$ when $\beta = 2^i$ (for some integer $i \ge 3$). This is done in Section 3. Section 2 gives some background materials. We conclude the paper with some remarks in Section 4.

2. BACKGROUND MATERIALS

We have the following results, due to Majumdar [1].

***Lemma 2.1* :** Let $\beta = 2^i$ (for some integer $i \ge 2$). Then, for all $n \ge 1$, $MT(n+1, \beta) - MT(n, \beta) = 2^s$ for some integer $s \ge 1$.

***Lemma 2.2* :** Let $\beta = 2^i$ (for some integer $i \ge 2$). Let $FT(N, k, \beta)$ be minimized at the two points $k = K, K+1$ for some integer $N \ge 1$. Let

$$M = 2N - K + i - 1.$$

Then, $FT(M, k, \beta)$ is minimized at the two values $k = N-1, N$.

***Lemma 2.3*:** Let, for some $N \ge 1$, $\beta \ge 2$, $FT(N, k, \beta)$ be minimized at the two points $k = K, K+1$. Then:

1. $FT(N-1, k, \beta)$ is minimized at $k = K$,
2. $FT(n+1, k, \beta)$ is minimized at $k = K+1$,
3. $MT(N, \beta) - MT(N-1, \beta) = 2^{N-K-1} = MT(N+1, \beta) - MT(N, \beta)$.

Lemma 2.4 below shows that, if $\beta = 2^i$ for some integer $i \ge 2$, there is an integer $N \ge 1$ such that $FT(N, k, \beta)$ is minimized at two values of k, and there is an integer $M \ge 1$ such that $FT(M, k, \beta)$ is minimized at a unique value of k.

***Lemma 2.4*:** Let $\beta = 2^i$ for some integer $i \ge 2$. Then,

1. $FT(i, k, \beta)$ is minimized at the point $k = 0$ only,
2. $FT(i+1, k, \beta)$ is minimized at the two points $k = 0, 1$; $FT(i+2, k, \beta)$ is minimized at the unique point $k = 1$.
3. $FT(i+3, k, \beta)$ is minimized at the two points $k = 1, 2$; $FT(i+4, k, \beta)$ is minimized at the unique point $k = 2$.

Proof: To prove part (1), note that:

$$FT(i, 0, \beta) \equiv 2^i - 1 < \beta\, MT(1, \beta) + 2^i - 1 \equiv FT(i+1, 1, \beta),$$

so, that by Lemma 1.4, $FT(i, k, \beta)$ is minimized at the unique point $k = 0$.

(2) Since

$$FT(i+1, 0, \beta) \equiv 2^{i+1} - 1 = \beta MT(1, \beta) + 2^i - 1 \equiv FT(i+1, 1, \beta),$$

by Lemma 1.4 and Lemma 1.3, $FT(i+1, k, \beta)$ is minimized at $k = 0, 1$. Again, since

$$FT(i+2, 0, \beta) \equiv 2^{i+2} - 1 > \beta MT(1, \beta) + 2^{i+1} - 1 \equiv FT(i+2, 1, \beta),$$

it follows that $FT(i+2, k, \beta)$ is minimized at the unique point $k = 1$.

(3) Note that:

$$FT(i+3, 1, \beta) \equiv \beta MT(1, \beta) + 2^{i+2} - 1 = \beta MT(2, \beta) + 2^{i+1} - 1 \equiv FT(i+3, 2, \beta).$$

Therefore, $FT(i+3, k, \beta)$ is minimized at $k = 1, 2$. Again, since

$$FT(i+4, 1, \beta) \equiv \beta MT(1, \beta) + 2^{i+3} - 1$$
$$> \beta MT(2, \beta) + 2^{i+2} - 1 \equiv FT(i+4, 2, \beta),$$

it follows that $FT(i+4, k, \beta)$ is minimized at the unique point $k = 2$.

Lemma 2.5: Let $\beta = 2^i$ for some integer $i \geq 2$. Let, for β fixed, $FT(N, k, \beta)$ be minimized at the unique point $k = K\ (\neq 0)$ for some integer $N \geq 1$. Then:

1. $FT(N+1, k, \beta)$ is minimized at $k = K$,
2. $FT(N-1, k, \beta)$ is minimized at the two points $k = K-1, K$.

Proof: By assumption,

$$MT(N, \beta) = \beta MT(K, \beta) + 2^{N-K} - 1 < \beta MT(K+1, \beta) + 2^{N-K-1} - 1,$$

$$MT(n, \beta) = \beta MT(K, \beta) + 2^{N-K} - 1 < \beta MT(K-1, \beta) + 2^{N-K+1} - 1,$$

and hence,

$$\beta[MT(K+1, \beta) - MT(K, \beta)] > 2^{N-K-1}, \qquad (1)$$

$$\beta[MT(K, \beta) - MT(K-1, \beta)] < 2^{N-K}. \qquad (2)$$

(1) If $FT(N+1, k, \beta)$ is not minimized at $k = K$, then by Lemma 1.3, $FT(N+1, k, \beta)$ is minimized at the unique point $k = K+1$, so that:

$$MT(N+1, \beta) = \beta MT(K+1, \beta) + 2^{N-K} - 1$$
$$< \beta MT(K, \beta) + 2^{N-K+1} - 1.$$

Thus,

$$\beta[MT(K+1, \beta) - MT(K, \beta)] < 2^{N-K},$$

which, together with the inequality (1), contradicts Lemma 2.1. Thus, $FT(N+1, k, \beta)$ is minimized at $k = K$. Then,

$$MT(N+1, \beta) = \beta MT(K, \beta) + 2^{N-K+1} - 1$$
$$\leq \beta MT(K+1, \beta) + 2^{N-K} - 1,$$

giving

$$\beta[MT(K+1, \beta) - MT(K, \beta)] \geq 2^{N-K}. \qquad (3)$$

Observe that,

$$MT(N+1, \beta) - MT(N, \beta) = 2^{N-K}. \qquad (2.1)$$

(2) If $FT(N-1, k, \beta)$ is not minimized at $k = K$, then by virtue of Lemma 1.3, it is minimized at the unique point $k = K-1$, so that

$$MT(N-1, \beta) = \beta MT(K-1, \beta) + 2^{N-K} - 1$$
$$< \beta MT(K, \beta) + 2^{N-K-1} - 1.$$

Thus,

$$\beta[MT(K, \beta) - MT(K-1, \beta)] > 2^{N-K-1},$$

which, together with the inequality (2), contradicts Lemma 2.1. Hence, $FT(N-1, k, \beta)$ is minimized at $k = K$. We now want to show that $FT(N-1, k, \beta)$ is minimized at $k = K-1$ as well, for otherwise;

$$MT(N-1, \beta) = \beta MT(K, \beta) + 2^{N-K-1} - 1$$
$$< \beta MT(K-1, \beta) + 2^{N-K} - 1,$$

giving

$$\beta[MT(K, \beta) - MT(K-1, \beta)] < 2^{N-K-1}.$$

Using the inequality (3), we get:

$$\beta[MT(K+1, \beta) - MT(K, \beta)] \geq 2^{N-K}$$
$$> 2\beta[MT(K, \beta) - MT(K-1, \beta)],$$

which contradicts Lemma 1.2. Hence, $FT(N-1, k, \beta)$ is minimized at $k = K-1$ as well.

Here,

$$MT(N-1, \beta) = \beta MT(K, \beta) + 2^{N-K-1} - 1$$
$$= \beta MT(K-1, \beta) + 2^{N-K} - 1,$$

so that

$$MT(N, \beta) - MT(N-1, \beta) = 2^{N-K-1}. \qquad (2.2)$$

Moreover,

$$MT(N, \beta) - MT(N-1, \beta) = \beta[MT(K, \beta) - MT(K-1, \beta)]. \qquad (2.3)$$

Corollary 2.1: Let $\beta = 2^i$ for some integer $i \geq 1$. Let, for β fixed, $FT(N, k, \beta)$ be minimized at the unique point $k = K$ for some integer $N \geq 1$. Then, $FT(N + 1, k, \beta)$ is minimized at the two points $k = K, K + 1$, with

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K}.$$

Proof **:** If $FT(N + 1, k, \beta)$ is minimized at the unique point $k = K$, then by part (b) of Lemma 2.5, $FT(N, k, \beta)$ is minimized at two values of k, contrary to the assumption. Thus,

$$MT(N + 1, \beta) = \beta MT(K, \beta) + 2^{N-K+1} - 1$$
$$= \beta MT(K + 1, \beta) + 2^{N-K} - 1,$$

which gives

$$\beta[MT(K + 1, \beta) - MT(K, \beta)] = 2^{N-K}$$
$$= MT(N + 1, \beta) - MT(N, \beta). \qquad (2.4)$$

Theorem 2.1: Let $\beta = 2^i$ for some integer $i \geq 2$. Let, for β fixed, $FT(N, k, \beta)$ be minimized at the point $k = K$ for some integer $N \geq 1$. Then,

$$MT(N, \beta) - MT(N-1, \beta) = 2^{N-K-1}.$$

Moreover,

1. if $FT(N, k, \beta)$ is minimized at the two points $k = K, K + 1$, then

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K-1}$$
$$= \beta[MT(K + 1, \beta) - MT(K, \beta)],$$

2. if $FT(N, k, \beta)$ is minimized at the unique point $k = K$, then

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K}$$
$$= \beta[MT(K + 1, \beta) - MT(K, \beta)].$$

Proof: follows readily from Lemma 2.3, (2.2) and (2.4).

Let $\beta = 2^i$ for some integer $i \geq 2$. Starting with a uniquely attained function $MT(N, \beta)$ (for β fixed), the lemma below finds another.

Lemma 2.6: Let $\beta = 2^i$ (for any integer $i \geq 2$). Let, for β fixed, $FT(N, k, \beta)$ be minimized at the unique point $k = K$. Let

$$M = 2N - K + i - 1. \qquad (2.5)$$

Then, $FT(M + 1, k, \beta)$ is minimized at the unique point $k = N$.

Proof **:** Since $FT(N, k, \beta)$ is minimized at the unique point $k = K$, $FT(N + 1, k, \beta)$ is minimized at the two points $k = K, K + 1$ (by Corollary 2.1). Therefore, using (2.5), from (2.1) and (2.3), we get

$$\beta[MT(N + 1, \beta) - MT(N, \beta)] = 2^{N-K+i} = 2^{M-N+1}, \qquad (2.6)$$

$$\beta[MT(N, \beta) - MT(N-1, \beta)] = 2^{N-K+i-1} = 2^{M-N}. \qquad (2.7)$$

By part (2) of Lemma 2.5, $FT(N-1, k, \beta)$ is minimized at the two points $k = K-1, K$, and so, by part (3) of Lemma 2.2,

$$MT(N-1, \beta) - MT(N-2, \beta) = 2^{N-K-1}.$$

Therefore, by (2.5),

$$\beta[MT(N-1, \beta) - MT(N-2, \beta)] = 2^{N-K+i-1} = 2^{M-N}. \qquad (2.8)$$

Now, if,

$$MT(M, \beta) = \beta MT(N-1, \beta) + 2^{M-N+1} - 1$$
$$= \beta MT(N-2, \beta) + 2^{M-N+2} - 1,$$

we get,

$$\beta[MT(N-1, \beta) - MT(N-2, \beta)] = 2^{N-K+i-1} = 2^{M-N+1},$$

contradicting (2.8). Thus, $FT(M, k, \beta)$ is not minimized at $k = N-2$. Again, if

$$MT(M, \beta) = \beta MT(N-1, \beta) + 2^{M-N+1} - 1$$
$$< \beta MT(N, \beta) + 2^{M-N} - 1,$$

we get

$$\beta[MT(N, \beta) - MT(N-1, \beta)] > 2^{M-N},$$

which contradicts (2.7). Hence, $FT(M, k, \beta)$ is minimized at the two points $k = N - 1, N$. Then, $FT(M + 1, k, \beta)$ is minimized at the unique point $k = N$, because otherwise,

$$MT(M + 1, \beta) = \beta MT(N, \beta) + 2^{M-N+1} - 1$$
$$= \beta MT(N + 1, \beta) + 2^{M-N} - 1,$$

so that,

$$\beta[MT(N + 1, \beta) - MT(N, \beta)] = 2^{M-N},$$

contradicting (2.6). Thus, $MT(M + 1, \beta)$ is attained at the unique point $k = N$.

***Lemma 2.7*:** Let $\beta = 2^i$ for some integer $i \geq 2$. Let, for β fixed, $FT(N, k, \beta)$ be minimized at the unique point $k = K$ for some integer $N \geq 1$, so that,

$$MT(N, \beta) - MT(N - 1, \beta) = 2^{N-K-1},$$

$$MT(N + 1, \beta) - MT(N, \beta) = 2^{N-K}.$$

Let,

$$M = min\{n : MT(n, \beta) - MT(n - 1, \beta) = 2^{N-K-1}\}.$$

Then, $FT(n, k, \beta)$ is minimized at two values of k, for any n satisfying $M \leq n \leq N - 1$.

***Proof*:** By assumption,

$$MT(M, \beta) - MT(M - 1, \beta) = 2^{N-K-1},$$

but

$$MT(M - 1, \beta) - MT(M - 2, \beta) = 2^{N-K-2}.$$

Let $FT(M, k, \beta)$ be minimized at $k = L$ (if $FT(M, k, \beta)$ is minimized at two values of k, L is the minimum of the two values). Then, from Theorem 2.1,

$$MT(M, \beta) - MT(M - 1, \beta) = 2^{M-L-1}.$$

Now, if $FT(M - 1, k, \beta)$ is minimized at the two points $k = L - 1, L$, then by part (3) of Lemma 2.3,

$$MT(M - 1, \beta) - MT(M - 2, \beta) = 2^{M-L-1},$$

which contradicts the definition of M. Consequently, $FT(M - 1, k, \beta)$ is minimized at the unique point $k = L$, and hence, by virtue of Lemma 2.5 and Corollary 2.1, $FT(M, k, \beta)$ is minimized at the two points $k = L, L + 1$, so that, by part (2) of Lemma 2.2, $FT(M + 1, k, \beta)$ is minimized at $k = L + 1$. Now, if $FT(M + 1, k, \beta)$ is minimized at the unique point $k = L + 1$, then by (2.2) and (2.1),

$$MT(M + 1, \beta) - MT(M, \beta) = 2^{M-L-1},$$
$$MT(M + 2, \beta) - MT(M + 1, \beta) = 2^{M-L},$$

and we must have $M + 1 = N$. Otherwise, $FT(M + 1, k, \beta)$ is minimized at two values of k, namely, at $k = L + 1, L + 2$. Continuing in this way, we see that each of the functions $FT(M, k, \beta)$, $FT(M + 1, k, \beta)$, ..., $FT(N - 1, k, \beta)$ is minimized at two values of k; more precisely, for $\ell = 0, 1, ..., FT(M + \ell, k, \beta)$ is minimized at the two points $k = L + \ell, L + \ell + 1$.

The next section considers the problem of finding the solution of the recurrence relation (1.1) when $\beta = 2^i$ for some integer $i \geq 2$.

3. THE SOLUTION OF THE RECURRENCE RELATION

This section derives a closed-form expression of $MT(n, \beta)$ when $\beta = 2^i$ for any integer $i \geq 2$.

For any $\beta \geq 2$ fixed, let:

$$a_n = MT(n, \beta) - MT(n - 1, \beta), \quad n \geq 1. \tag{3.1}$$

Let $\beta = 2^i$ for some integer $i \geq 2$. Let k_j ($j \geq 0$) be the largest index such that

$$a_{k_j} = 2^j. \tag{3.2}$$

It may be noted here that, when $\beta = 2^i$ for some integer $i \geq 2$, and k_j is the largest index satisfying the condition (3.2), then $MT(k_j, \beta)$ is attained at a unique point, for otherwise, if $MT(k_j, \beta)$ is attained at $k = K, K+1$, then by part (3) of Lemma 2.3, the definition of k_j is violated.

Clearly, for any $\beta \geq 2$,

$$k_0 = 1,\ k_1 = 2.$$

However, the sequence of numbers $\{k_n\}_{n\geq 0}$, $n \geq 3$, depends on β. For example,

$$k_2(\beta) = \begin{cases} 4, & if \quad \beta = 3, 4 \\ 3, & if \quad \beta \geq 5 \end{cases}$$

as can easily be verified.

***Theorem 3.1*:** Let $\beta = 2^i$ for some integer $i \geq 2$. Let k_j $(j \geq 0)$ be the largest index such that

$$a_{k_j} = 2^j.$$

Then,

1. $MT(k_j, \beta)$ is attained uniquely at the point $k = k_j - j - 1$,
2. $MT(n, \beta)$ is attained at the two points $k = n - j - 1$, $n - j - 2$ for all n with $k_j + 1 \leq n \leq k_{j+1} - 1$,
3. $MT(k_{j+1}, \beta)$ is attained at the point $k = k_{j+1} - j - 2$.

Proof: We prove the theorem by induction on j. The results can easily be verified when $j = 0$ and $j = 1$. So, we assume that the results are true for some j.

Now, in the notation of Lemma 2.7,

$$N = k_j,\ N - K - 1 = j,\ N + M = k_{j+1}.$$

1. To prove part (1), note that $MT(k_j, \beta)$ is attained at the unique point $k = K = k_j - j - 1$.
2. follows immediately from Lemma 2.7 with $M = k_j + 1$, $L = M - j - 1 = k_j - j$, $N = k_{j+1}$, so that $MT(M + \ell, \beta)$ is attained at the two points $k = M + \ell - j - 1$, $M + \ell - j$.
3. Since (by part (2) above), $MT(k_{j+1} - 1, \beta)$ is attained at the two points;

$$k = k_{j+1} - j - 3,\ k_{j+1} - j - 2,$$

it follows, by Lemma 2.3, that $MT(k_{j+1}, \beta)$ is attained (uniquely) at the point $k = k_{j+1} - j - 2$. Thus, the results are true for $j + 1$ as well, completing induction.

When $\beta = 2^i$ for some integer $i \geq 1$, an expression of $MT(n, \beta)$ in terms of the numbers k_j can be derived. This is done in the following theorem. We then illustrate the use of Theorem 3.2 by finding $MR\left(\frac{(j+1)(j+2)}{2}\right)$ corresponding to the Reve`s puzzle in Lemma 3.1.

***Theorem 3.2*:** Let $\beta = 2^i$ for some integer $i \geq 1$. Let k_j $(j \geq 0)$ be the largest index such that;

$$a_{k_j} = 2^j.$$

Let $k_j \leq n < k_{j+1}$ for some $j \geq 0$. Then,

$$MT(n, \beta) = 1 + \sum_{\ell=1}^{j} (k_\ell - k_{\ell-1}) 2^\ell + 2^{j+1}(n - k_j).$$

***Proof* :** We write $MT(n, \beta)$ as follows :

$$MT(n, \beta) = \sum_{m=1}^{n} [MT(m, \beta) - MT(m-1, \beta)]$$

$$= 1 + \sum_{\ell=1}^{j} \sum_{m=k_{\ell-1}+1}^{k_\ell} [MT(m, \beta) - MT(m-1, \beta)]$$

$$+ \sum_{m=k_j+1}^{n} [MT(m, \beta) - MT(m-1, \beta)].$$

Now, noting that

$MT(m, \beta) - MT(m-1, \beta) = 2^{\ell}$ if $k_{\ell-1} + 1 \le m \le k_\ell$; $\ell = 1, 2, \ldots,$

the desired expression follows.

Recall that $\beta = 2$ correspond to the Reve's puzzle.

Let $MR(n)$ is the minimum number of moves required to solve the Reve's puzzle with $n \geq 1$ discs. It is well-known (see, for example, Roth [4], Hinz [5] and Majumdar [6, 7]) that, for $j \geq 0$, $MR\left(\frac{(j+1)(j+2)}{2}\right)$ is attained at the unique point $k = \frac{j(j+1)}{2}$. We now give an expression of $MT(\frac{(j+1)(j+2)}{2}, 2) = MR\left(\frac{(j+1)(j+2)}{2}\right)$, $j \geq 0$, in the following lemma, which makes use of Theorem 3.2.

***Lemma 3.1*:** For $j \geq 0$,

$$MT\left(\frac{(j+1)(j+2)}{2}, 2\right) = MR\left(\frac{(j+1)(j+2)}{2}\right) = j2^{j+1} + 1.$$

Proof **:** We first note that, when $\beta = 2$, k_ℓ of Theorem 3.2 is given by

$$k_\ell = \frac{(\ell+1)(\ell+2)}{2} \quad \text{for all } \ell \geq 0.$$

Therefore, by Theorem 3.2,

$$MR\left(\frac{(j+1)(j+2)}{2}\right) = 1 + \sum_{\ell=1}^{j}\left[\frac{(\ell+1)(\ell+2)}{2} - \frac{\ell(\ell+1)}{2}\right]2^{\ell}$$
$$= 1 + \sum_{\ell=1}^{j}(\ell+1)2^{\ell}.$$

Now, let

$$S = \sum_{\ell=1}^{j}(\ell+1)2^{\ell} = \sum_{\ell=1}^{j}\ell 2^{\ell} + 2(2^j - 1).$$

Then,

$$2S = \sum_{\ell=1}^{j}(\ell+1)2^{\ell+1} = \sum_{\ell=1}^{j-1}(\ell+1)2^{\ell+1} + (j+1)2^{j+1}$$
$$= \sum_{k=2}^{j}k2^{k} + (j+1)2^{j+1}$$
$$= \left(\sum_{\ell=1}^{j}\ell 2^{\ell} - 2\right) + (j+1)2^{j+1}$$
$$= S - 2^{j+1} + (j+1)2^{j+1},$$

so that

$$S = j2^{j+1}.$$

Hence, finally we get the desired expression.

The expression of $MR(\frac{(j+1)(j+2)}{2})$ is well-known, and can be found in, for example, Roth [4], Hinz [5] and Majumdar [6, 7]). Lemma 3.1 gives an alternative approach to find it. We now state and prove the following theorem.

***Theorem 3.3*:** Let $\beta = 2^i$ for some integer $i \geq 2$. Then, for any $n \geq 0$, $MT\left((n+1)(\frac{i}{2}n + \ell), \beta\right)$ is attained at the unique point $k = n\left[\frac{i}{2}(n-1) + \ell\right]$, $1 \le \ell \le i$.

***Proof*:** The proof is by induction on n.

Noting that $MT(n, \beta)$ is attained at the unique point $k = 0$ for all $1 \le n \le i$ (by virtue of part (1) of Lemma 2.4), the validity of the result for $n = 0$ follows. So, we assume that the result holds true for some n. Then, by Lemma 2.6, $MT(N+1, \beta)$ is attained at the unique point $k = (n+1)(\frac{i}{2}n + \ell)$, where;

$$N + 1 = 2(n+1)(\tfrac{i}{2}n + \ell) - n[\tfrac{i}{2}(n-1) + \ell] + i$$
$$= (n+2)[\tfrac{i}{2}(n+1) + \ell].$$

This shows that the result is true for $n + 1$ as well, thereby completing induction.

The following two lemmas deal respectively with the particular cases when $\beta = 4$ and $\beta = 8$.

***Lemma 3.2*:** For all $n \geq 1$,

1. $MT(n^2, 4) = \frac{1}{3}[(3n-2)2^{2n-1}+1]$,
2. $MT(n(n+1), 4) = \frac{1}{3}[(3n-1)2^{2n}+1]$.

Proof: By Theorem 3.3, $MT(n^2, 4)$ is attained at the (unique) point $k = (n-1)^2$.

Therefore,

$$MT(n^2, 4) = 4\ MT((n-1)^2, 4) + 2^{2n-1} - 1$$

$$= 4[4\ MT((n-2)^2, 4) + 2^{2n-3} - 1] + 2^{2n-1} - 1$$

$$= 4^2 MT((n-2)^2, 4) + 2.2^{2n-1} - (1+4).$$

Continuing in this way ℓ times, we get:

$$MT(n^2, 4)$$
$$= 4^{\ell}\ MT((n-\ell)^2, 4) + \ell.2^{2n-1} - (1+4+...+4^{\ell-1})$$

$$= 4^{\ell}\ MT((n-\ell)^2, 4) + \ell.2^{2n-1} - \frac{4^{\ell}-1}{3}.$$

Now, choosing $\ell = n$ and then simplifying, we get the expression desired.

(2) Since $MT(n(n+1),4)$ is attained at $k = n(n-1)$, by repeated application, we get

$$MT(n(n+1), 4) = 4\ MT(n(n-1), 4) + 2^{2n} - 1$$

$$= 4[4\ MT((n-1)(n-2), 4) + 2^{2n-2} - 1] + 2^{2n} - 1$$

$$= 4^2 MT((n-1)(n-2), 4) + 2.2^{2n} - (1+4).$$

After ℓ iterations, we get

$$MT(n(n+1), 4)$$
$$= 4^{\ell} MT((n-\ell+1)(n-\ell), 4) + \ell.2^{2n} - (1+4+...+4^{\ell-1})$$

$$= 4^{\ell}\ MT((n-\ell+1)(n-\ell), 4) + \ell.2^{2n} - \frac{4^{\ell}-1}{3}.$$

Finally, putting $\ell = n$, we get the desired result after simplification.

***Corollary 3.1*:** For any integer $n \geq 1$,

$$MT(N, 4) = \begin{cases} \frac{(3n-2)2^{2n-1}+1}{3} + (N-n^2)2^{2n-1}, & if\ \ n^2 \leq N \leq n(n+1) \\ \frac{(3n-1)2^{2n}+1}{3} + [N-n(n+1)]2^{2n}, & if\ \ n(n+1) \leq N < (n+1)^2 \end{cases}$$

***Proof*:** Since

$$MT(n^2+m, 4) - MT(n^2+m-1, 4) = 2^{2n-1} \text{ for all } 1 \leq m \leq n,$$

$$MT(n(n+1)+m, 4) - MT(n(n+1)+m-1, 4) = 2^{2n}$$

for all $1 \leq m \leq n+1$,

the result follows from Lemma 3.2.

***Lemma 3.3*:** For any integer $n \geq 1$,

1. $MT\left(\frac{3}{2}n(n+1), 8\right) = \frac{1}{7}\left[(7n-1)2^{3n}+1\right]$,
2. $MT\left(\frac{(n+1)(3n+2)}{2}, 8\right) = \frac{1}{7}\left[(7n+3)2^{3n+1}+1\right]$,
3. $MT\left(\frac{(n+1)(3n+4)}{2}, 8\right) = \frac{1}{7}\left[(7n+5)2^{3n+2}+1\right]$.

Proof: The proofs are given below.

(1) By Theorem 3.3, $MT(\frac{3}{2}n(n+1), 8)$ is attained at the point $k = \frac{3}{2}n(n-1)$, so that

$$MT(\tfrac{3}{2}n(n+1), 8) = 8\,MT(\tfrac{3}{2}n(n-1), 8) + 2^{3n} - 1$$

$$= 8[8MT(\tfrac{3}{2}(n-1)(n-2), 8) + 2^{3(n-1)} - 1] + 2^{3n} - 1$$

$$= 8^2 MT(\tfrac{3}{2}(n-1)(n-2), 8) + 2.2^{3n} - (1+8).$$

Continuing ℓ times, we get

$$MT(\tfrac{3}{2}n(n+1), 8) = 8^{\ell} MT(\tfrac{3}{2}(n-\ell+1)(n-\ell), 8)$$
$$+ \ell.2^{3n} - (1+8+...+8^{\ell-1})$$

$$= 8^{\ell} MT(\tfrac{3}{2}(n-\ell+1)(n-\ell), 8) + \ell.2^{3n} - \tfrac{1}{7}(8^{\ell}-1).$$

In the above expression, putting $\ell = n$, and then simplifying, we get the desired expression.

(2) Since $MT(\frac{(n+1)(3n+2)}{2}, 8)$ is attained at the point $k = \frac{n(3n-1)}{2}$, we get

$$MT(\frac{(n+1)(3n+2)}{2}, 8) = 8\,MT(\frac{n(3n-1)}{2}, 8) + 2^{3n+1} - 1$$

$$= 8[8MT(\frac{(n-1)(3n-4)}{2}, 8) + 2^{3n-2} - 1] + 2^{3n+1} - 1$$

$$= 8^2 MT(\frac{(n-1)(3n-4)}{2}, 8) + 2.2^{3n+1} - (1+8).$$

And in general, after ℓ iterations, we have:

$$MT(\frac{(n+1)(3n+2)}{2}, 8) = 8^{\ell} MT(\tfrac{1}{2}(n-\ell+1)(3(n-\ell) + 2), 8) + \ell.2^{3n+1} - (1+8+ \ldots + 8^{\ell-1})$$

$$= 8^{\ell} MT(\tfrac{1}{2}(n-\ell+1)(3(n-\ell)+2), 8) + \ell.2^{3n+1} - \tfrac{1}{7}(8^{\ell}-1).$$

Now, putting $\ell = n$, we get

$$MT(\frac{(n+1)(3n+2)}{2}, 8) = 8^n + n.2^{3n+1} - \tfrac{1}{7}(8^n - 1).$$

Simplifying, we get the result desired.

(3) Since $MT(\frac{(n+1)(3n+4)}{2}, 8)$ is attained at the point $k = \frac{n(3n+1)}{2}$, we have

$$MT(\frac{(n+1)(3n+4)}{2}, 8) = 8\,MT(\frac{n(3n+1)}{2}, 8) + 2^{3n+2} - 1$$

$$= 8[8MT(\frac{(n-1)(3n-2)}{2}, 8) + 2^{3n-1} - 1] + 2^{3n+2} - 1$$

$$= 8^2 MT(\frac{(n-1)(3n-2)}{2}, 8) + 2.2^{3n+2} - (1+8),$$

and after ℓ iterations, we have:

$$MT(\frac{(n+1)(3n+4)}{2}, 8) = 8^{\ell} MT(\tfrac{1}{2}(n-\ell+1)(3(n-\ell) + 4), 8) + \ell.2^{3n+2} - (1+8+ \ldots + 8^{\ell-1})$$

$$= 8^{\ell} MT(\tfrac{1}{2}(n-\ell+1)(3(n-\ell)+4), 8) + \ell.2^{3n+1} - \tfrac{1}{7}(8^{\ell}-1).$$

Finally, putting $\ell = n$, we get

$$MT(\frac{(n+1)(3n+4)}{2}, 8) = 3.8^n + n.2^{3n+2} - \tfrac{1}{7}(8^n - 1),$$

which gives the desired result after simplification.

4. REMARKS

In this paper, we derive some results in connection with the difference $MT(n+1, \beta) - MT(n, \beta)$, which plays a vital role in solving the recurrence relation (1.1). These are given in Section 2. In Section 3, an alternative expression of $MT(n, \beta)$ is given when $\beta = 2^i$ for some integer $i \geq 1$. From Theorem 3.2, we observe that the determination of the numbers k_j, satisfying the condition, $MT(k_j, \beta) - MT(k_j - 1, \beta) = 2^j$, is required, which is given in Theorem 3.3. This would enable us to find a closed-form expression of $MT(n, \beta)$, as has been illustrated in Lemma 3.2 and Corollary 3.1 explicitly for the particular case when $\alpha = 4$.

It may be mentioned here that, Matsuura [2] adopted a different approach to find $MT(n, \beta)$. More specifically, letting $\{b_n\}_{n\geq 1}$ be the sequence of numbers defined as follows:

$$b_n = 2^m \beta^{\ell};\ m \geq 0,\ \ell \geq 0,$$

and arranged in non-decreasing order, Matsuura [2] showed, by induction on n, using a recurrence relation satisfied by b_n, that $a_n = b_n$. In this paper, we follow a different approach, which enables us to find an explicit form of $MT(n, \beta)$. Our analysis reveals many interesting properties and local-value relationships that are inherent in the optimal value function $MT(n, \beta)$. Moreover, though for small values of n, b_n may be found out, for large values of n, finding b_n, even using the recurrence relation satisfied by it, is a challenging problem.

For $\alpha = 4$, the first few terms of the sequence $\{b_n\}_{n\geq 1}$ are:

1, 2, 4, 4, 8, 8, 16, 16, 16, 32, 32, 32, 64, ….

We recall that, when $\beta = 2$, $MT(n, \beta)$ satisfies exactly one of the following relationships :

$$MT(n+2, \beta) - MT(n+1, \beta) = 2[MT(n+1, \beta) - MT(n, \beta)], \quad (4.1)$$

$$MT(n+2, \beta) - MT(n+1, \beta) = MT(n+1, \beta) - MT(n, \beta). \quad (4.2)$$

When $\beta = 2^i$ for some integer $i \geq 2$, we have the following results.

***Lemma 4.1*:** Let $\beta = 2^i$ (for some integer $i \geq 1$). Then, $MT(n, \beta)$ satisfies the relationship (4.1) for some integer $n \geq 1$ if and only if $MT(n+1, \beta)$ is attained at a unique value of k.

Proof: If $MT(n+1, \beta)$ is attained at the unique point $k = K$, then by Theorem 2.1,

$$MT(n+2, \beta) - MT(n+1, \beta) = 2^{n-K+1}$$
$$= 2[MT(n+1, \beta) - MT(n, \beta)].$$

Conversely, let the relationship (4.1) hold. Now, if, $MT(n+1, \beta)$ is attained at the two points $k = K, K+1$, then by Theorem 2.1,

$$MT(n+2, \beta) - MT(n+1, \beta) = 2^{n-K}$$
$$= MT(n+1, \beta) - MT(n, \beta),$$

which is in contradiction with the assumption.

***Corollary 4.1*:** Let $\beta = 2^i$ (for some integer $i \geq 1$). Then, $MT(n, \beta)$ satisfies the relationship (4.2) for some $n \geq 1$ if and only if $MT(n+1, \beta)$ is attained at two values of k.

It has been proved that, when $\beta \neq 2^i$ for any integer $i \geq 1$, $MT(n, \beta)$ is attained at a unique value of k (see Corollary 3.5 in Majumdar [1]). It then follows, by Corollary 4.1 above that, in such a case, $MT(n, \beta)$ does not satisfy the relation (4.2).

***Corollary 4.2*:** For $\beta = 2^i$ (for some integer $i \geq 1$), the relationship (4.1) cannot hold for all $n \geq 1$.

***Proof*:** Let $MT(n, \beta)$, $MT(n+1, \beta)$ and $MT(n+2, \beta)$ satisfy the relationship (4.1). By Lemma 4.1, $MT(n+1, \beta)$ is attained at a unique point, say, $k = K$. Then, by Corollary 2.1, $MT(n+2, \beta)$ is attained at the points $k = K, K+1$, so that by Theorem 2.1,

$$MT(n+3, \beta) - MT(n+2, \beta) = 2^{n-K+1}$$
$$= MT(n+2, \beta) - MT(n+1, \beta).$$

This shows that, $MT(n+1, \beta)$, $MT(n+2, \beta)$ and $MT(n+3, \beta)$ satisfy the relationship (4.2).

***Lemma 4.2* :** Let $\beta = 2^i$ (for some integer $i \geq 1$). Then, $MT(n+1, \beta)$ is attained at the unique point $k = K$ (for some integer $n \geq 1$) if and only if $MT(K, \beta)$ is attained at a unique value of k.

***Proof* :** Let $MT(n+1, \beta)$ be attained at the unique point $k = K$. Then, by Lemma 4.1, $MT(n+1, \beta)$ satisfies the relationship (4.1), so that by (2.3) and (2.4),

$$MT(K+1, \beta) - MT(K, \beta) = 2[MT(K, \beta) - MT(K-1, \beta)],$$

so that, $MT(K, \beta)$ is attained at a unique value of k. Next, let $MT(K, \beta)$ be attained at a unique k. Then, by Lemma

2.6, we can find $MT(n + 1, \beta)$ which is attained at the unique point $k = K$.

In Table 4.1 and Table 4.2, we give the values of $MT(n, \beta)$ for $1 \leq n \leq 19$, $\beta = 4, 8, 16$.

Table 1. Values of $MT(n, \beta)$ for $1 \leq n \leq 10$ and $\beta = 4, 8, 16$.
In each cell, the number in parenthesis gives the value(s) of k at which $MT(n, \beta)$ is attained.

α \ n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4	1	3	7	11	19	27	43	59	75	107
	(0)	(0)	(0, 1)	(1)	(1, 2)	(2)	(2, 3)	(3, 4)	(4)	(4, 5)
8	1	3	7	15	23	39	55	87	119	183
	(0)	(0)	(0)	(0, 1)	(1)	(1, 2)	(2)	(2, 3)	(3)	(3, 4)
16	1	3	7	15	31	47	79	111	175	239
	(0)	(0)	(0)	(0)	(0, 1)	(1)	(1, 2)	(2)	(2, 3)	(3)

Table 2. Values of $MT(n, \beta)$ for $11 \leq n \leq 19$ and $\beta = 4, 8, 16$.

In each cell, the number in parenthesis gives the value(s) of k at which MT(n, β) is attained

α \ n	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
4	139	171	235	299	363	427	555	683	811
	(5, 6)	(6)	(6, 7)	(7, 8)	(8, 9)	(9)	(9, 10)	(10, 11)	(11, 12)
8	247	311	439	567	695	951	1207	1463	1975
	(4, 5)	(5)	(5, 6)	(6, 7)	(7)	(7, 8)	(8, 9)	(9)	(9, 10)
16	367	495	751	1007	1263	1775	2287	2799	3823
	(3, 4)	(4)	(4, 5)	(5, 6)	(6)	(6, 7)	(7, 8)	(8)	(8, 9)

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