Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences: 57 (2): 35-40 (2020)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

Sixth Order Numerical Iterated Method of Open Methods for Solving Nonlinear Applications Problems

Umair Khalid Qureshi[1*], Zubair Ahmed kalhoro[2], Asif Ali Shaikh[3], and Sanaullah Jamali[4]

[1]Department of Business Administration, Shaheed Benazir Bhutto University, Sanghar, Sindh, Pakistan
[2]Institute Mathematics and Computer Science, University of Sindh Jamshoro, Pakistan
[3]Department of Basic Sciences and Related Sciences, Mehran University of Engineering and Technology, Jamshoro, Sindh, Pakistan.
[4]Department of Mathmatics, University of Sindh Laa, Badin, Sindh, Pakistan

Abstract: This paper aims to construct a numerical iterated method of open methods to find a single root of application problems. The proposed numerical iterated method is the sixth order of convergence, and which is based on Steffensen Method and Newton Raphson Method. The proposed sixth-order numerical iterated method is compared with the Modified Efficient Iterative Method and Generalize Newton Raphson Method [16-17]. C++/MATLAB is used on a few examples for justification of the proposed method based on the number of evolutions, accuracy, and iterations. From numerical results, it has been observed that the sixth order numerical iterated method is good accuracy with good convergence criteria as the assessment of existing methods for solving the root of nonlinear applications problems.

Keywords: Non-linear application problems, Open methods, Sixth order methods, Convergence analysis.

1. INTRODUCTION

For estimating a root of nonlinear equations has wide applications in numerous branches of pure science and applied science has been deliberated in the general framework [1-5], such as non-linear equations:

$$f(x)=0$$

Due to the importance of $f(x)=0$, greatest researchers and scientists have been taken attention and lots of variants of accelerated methods had been given by using different techniques, such as Taylor series, quadrature formulas, homotopy perturbation method, Adomian's decomposition, and variationally iteration technique [6-10]. Similarly, Newton Raphson method is an important and basic method, which is fast converging numerical techniques but are not reliable because keeping a kind of pitfall, however, it converges quadratically [11-12], such as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f`(x_n)}$$

Furthermore, in numerical analysis order of convergence and computational efficiency become an efficient way to obtain the approximate solution and it has been a vigorous field of research. Whereas Order of convergence is a speed at which a given iterated sequence converges to the root and computational efficiency show the economy of the iterated method. To improve the order of convergence and computational efficiency numerous methods have been proposed with the help of the Newton Raphson method [13-15]. Similarly, in this paper, a sixth-order iterated method has been recommended for finding a single root of a nonlinear equation. The proposed sixth-order iterated method is a combination of the classical Steffensen method and the Newton Raphson Method. The proposed method has been compared with Modified Efficient

Received: March 2020; Accepted: June 2020
*Corresponding Author: Umair Khalid <umair.khalid_sng@sbbusba.edu.pk>

Iterative Method [16]:

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$
$$z_n = x_n - \frac{3f`(x_n) - f`(y_n)f(y_n)}{f`(x_n) + f`(y_n)f`(x_n)}$$
$$x_{n+1} = z_n + \frac{3f`(x_n) + f`(y_n)f(z_n)}{f`(x_n) - f`(y_n)f`(x_n)}$$

and Generalize Newton Raphson Method [17]

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$
$$x_{n+1} = y_n - \frac{f'(y_n) - \sqrt{f'(y_n)^2 - 2f''(y_n)f(y_n)}}{f''(y_n)}$$

The sixth order numerical iterated method of open methods is performing well and more competent in approaching the root of nonlinear applications equations.

2. PROPOSED METHODOLOGY

The idea of the proposed iterated method comes from different references [4, 10, 11, 15, 16]. The proposed iterated method is developed with the help of point-slope form, numerical technique, and open methods. Consider the non-linear equation

$$f(x)=0$$

where `x` be a root of it and f is a function on under consideration interval containing `x`, and we suppose that $|f(x)| > 0$ for the root of a nonlinear equation. By using the point-slope form,

$$f(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$

To find the root of this line, the value of x such that f(x)=0 by solving the following equation for x:

$$0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$

For the solution of `x` than the above equation become as

$$x = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}f(x_0)$$

In general,

$$x_{n+3} = x_{n+1} - \frac{(x_{n+2} - x_{n+1})}{f(x_{n+2}) - f(x_{n+1})}f(x_{n+1}) \qquad (2)$$

Where

$$x_{n+2} = x_{n+1} + h \qquad (3)$$

By using numerical condition such as $h = \Delta(x_{n+1}) = f(x_{n+1})$ in (3), then substitute in (2), we get

or

$$x = x_{n+1} - \frac{x_{n+1} + f(x_{n+1}) - x_{n+1}}{f\big(x_{n+1} + f(x_{n+1})\big) - f(x_{n+1})}f(x_{n+1})$$

$$x = x_{n+1} - \frac{f(x_{n+1})}{f\big(x_{n+1} + f(x_{n+1})\big) - f(x_{n+1})}f(x_{n+1}) \qquad (4)$$

Where $x_{n+1} = x_n - \frac{f(x_n)}{f`(x_n)}$ to substitute in (4), we get

$$x = x_n - \frac{f(x_n)}{f`(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)^2}{f\left(x_n - \frac{f(x_n)}{f`(x_n)} + f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)\right) - f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)}$$

Finally, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f`(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)^2}{f\left(x_n - \frac{f(x_n)}{f`(x_n)} + f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)\right) - f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)} \qquad (5)$$

Henceforth, (5) is a six order of convergence iterated

3. CONVERGENCE CRITERIA

The following section will be shown that the Numerical Iterated Method is keeping six order of convergence, such as:

$$e_{n+1} = 3c^5 e_n{}^6 + o(e_n{}^7)$$

Proof:

Using the relation $e_n = x_n - a$ in Taylor series, therefore from Taylor series we estimate $f(x_n)$, $f`(x_n)$ and $f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)$ with using this condition c= (f ``(a)) /(2f`(a)) and ignoring higher-order term for easy to solve, such as:

$$f(x_n) = f`(a)(e_n + ce_n{}^2) \qquad --- (i)$$
$$f`(x_n) = f`(a)(1 + 2ce_n) \qquad --- (ii)$$

By using (*i*) and (*ii*) in (2), we get:

$$x_n - \frac{f(x_n)}{f`(x_n)} = e_n - \frac{e_n f`(a) + \frac{e^2{}_n f``(a)}{2}}{f`(a) + 2e_n \frac{f``(a)}{2}}$$

$$x_n - \frac{f(x_n)}{f`(x_n)} = e_n - e_n(1 + ce_n)(1 + 2ce_n)^{-1}$$

$$x_n - \frac{f(x_n)}{f`(x_n)} = e_n - e_n(1 + ce_n)(1 - 2ce_n)$$

$$x_n - \frac{f(x_n)}{f`(x_n)} = e_n - e_n(1 - ce_n)$$

$$x_n - \frac{f(x_n)}{f`(x_n)} = ce^2{}_n \qquad --- (iii)$$

Thus,

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right) = f`(a)(ce^2{}_n + c^3 e_n{}^4)$$

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right) = ce^2{}_n f`(a)(1 + c^2 e_n{}^2)$$

Finally, we get

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right) = ce^2{}_n f`(a)(1 + c^2 e_n{}^2) \qquad --- (iv)$$

Or

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)} + f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)\right) = f`(a)\left(ce^2{}_n + ce^2{}_n f`(a) + c^3 e_n{}^4\big(1 + 2f`(a)\big)\right) \qquad --- (v)$$

By using (iv) and (v), we get:

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)} + f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)\right) - f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right) = f`(a)(ce^2{}_n + ce^2{}_n f`(a) + c^3 e_n{}^4\big(1 + 2f`(a)\big) - ce^2{}_n - c^3 e_n{}^4))$$

$$f\left(x_n - \frac{f(x_n)}{f`(x_n)} + f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right)\right) - f\left(x_n - \frac{f(x_n)}{f`(x_n)}\right) = ce^2{}_n f`^2(a)(1 + 2c^2 e_n{}^2) \qquad --- (vi)$$

By using (iii), (iv) and (vi) in the developed method, we get

$$= ce^2{}_n - \frac{ce^2{}_n f`(a)(1 + c^2 e_n{}^2) ce^2{}_n f`(a)(1 + c^2 e_n{}^2)}{ce^2{}_n f`^2(a)(1 + 2c^2 e_n{}^2)}$$

$$e_{n+1} = ce^2{}_n - \frac{ce^2{}_n(1 + 2c^2 e_n{}^2 + c^4 e_n{}^4)}{(1 + 2c^2 e_n{}^2)}$$

$$e_{n+1} = ce^2{}_n - ce^2{}_n(1 + 2c^2 e_n{}^2 + c^4 e_n{}^4)(1 + 2c^2 e_n{}^2)^{-1}$$

$$e_{n+1} = ce^2{}_n - ce^2{}_n(1 + 2c^2 e_n{}^2 + c^4 e_n{}^4)(1 - 2c^2 e_n{}^2)$$

$$e_{n+1} = ce^2{}_n - ce^2{}_n(1 + 2c^2 e_n{}^2 + c^4 e_n{}^4 - 2c^2 e_n{}^2 - 4c^4 e_n{}^4)$$

$$e_{n+1} = ce^2{}_n - ce^2{}_n(1 - 3c^4 e_n{}^4)$$

$$e_{n+1} = 3c^5 e_n{}^6 + o(e_n{}^7)$$

Henceforth, it has been proven that the proposed numerical iterated method is the sixth order of convergence.

4. NUMERICAL OUTCOMES

This section proposed sixth-order iterated method is applied to a few physical applications functions to illustrate the efficiency of the developed method, such as:

i. $f(x) = Sinx - x + 1$ with $x = 2$ (Mass of the jumper)
ii. $f(x) = e^x - 4x$ with $x = 0.5$ (Volume of the gas depends on the temperatures)
iii. $f(x)$ = ex + x - 20 with $x = 2.5$ (Anti-symmetric buckling of a beam)
iv. $f(x) = x^3 - 9x + 1$ with $x = 0$ (Projectile motion of any system)
v. $f(x) = 2x - lnx - 7$ with $x = 4$ (Pollutant bacteria concentration)

The developed method is equated with the Modified Efficient Iterative Method [16]

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$

$$z_n = x_n - \frac{3f`(x_n) - f`(y_n)f(y_n)}{f`(x_n) + f`(y_n)f`(x_n)}$$

$$x_{n+1} = z_n + \frac{3f`(x_n) + f`(y_n)f(z_n)}{f`(x_n) - f`(y_n)f`(x_n)}$$

and Generalize Newton Raphson Method [17]

$$y_n = x_n - \frac{f(x_n)}{f`(x_n)}$$

$$x_{n+1} = y_n - \frac{f'(y_n) - \sqrt{f'(y_n)^2 - 2f''(y_n)f(y_n)}}{f''(y_n)}$$

C++/MATLAB is used to examine the proposed iterative method as shown in Table 1.

5. CONCLUSIONS

This study has been considered and analyzed as a Numerical Iterated Method of open methods for solving nonlinear physical application problems. The order of convergence of the Proposed Iterated

Table 1 Iterative method using C++/MATLAB

$S\#$	$f(x)$	Methods	I	Total function evolutions	x	AE%
1	$Sinx - x + 1 = 0$ $xo = 2$	Modified Efficient Iterative Method	3	15	1.93456	3.81470e-006
		Generalize Newton Raphson Method	2	10		3.81470e-006
		Numerical Iterated Method	2	8		3.81470e-006
2	$e^x - 4x = 0$ $xo = 0.5$	Generalize Newton Raphson Method	2	10	0.357403	5.96046e-008
		Modified Efficient Iterative Method	3	15		6.04987e-006
		Numerical Iterated Method	2	8		5.96046e-008
3	$e^x + x - 20 = 0$ $xo = 2.5$	Modified Efficient Iterative Method	3	15	2.84244	9.53674e-007
		Generalize Newton Raphson Method	2	10		9.53674e-007
		Numerical Iterated Method	2	8		9.53674e-007
4	$x^3 - 9x + 1 = 0$ $xo = 0$	Generalize Newton Raphson Method	2	10	0.111264	1.21012e-010
		Modified Efficient Iterative Method	3	15		7.45058e-007
		Numerical Iterated Method	2	8		1.56462e-007
5	$2x - lnx - 7 = 0$ $xo = 4$	Modified Efficient Iterative Method	3	15	4.21991	3.8147e-006
		Generalize Newton Raphson Method	2	10		3.8147e-006
		Numerical Iterated Method	2	8		3.8147e-006

Method is six, and it has derived from Steffensen Method and Newton Raphson Method. From the fallouts in Table-1, it has been experiential that the developed technique is reducing the number of evolution, good in accuracy as well as iteration lookout by the assessment of the Modified Efficient Iterative Method; on the other hand, the Generalize Newton Raphson Method gives equally accuracy and number of iterations but good in the evolution with the assessment of the proposed method. Hence, throughout the research study, it has been observed that the proposed sixth order method iterative method is decent execution, more competent and performing supercilious as the assessment of existing six order iterated methods for solving the single root of physical application nonlinear problems.

6. ACKNOWLEDGEMENTS

I would like to thank Prof. Dr. Asif Ali Shaikh, Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology Jamshoro, Sindh, Pakistan and faculty member of Shaheed Benazir Bhutto University, Sanghar, Sindh, Pakistan for encouraging my research and for allowing me to grow as a researcher. Finally, a special thanks to my family for supporting me everywhere at any stage.

7. REFERENCES

1. N. Yasmin, and M.U.D. Junjua, Some Derivative-Free Iterative Methods for Solving Nonlinear Equations, *Academic Research International.* 2(1) 75-82 (2012).

2. C.N. Iwetan, I.A. Fuwape, M.S. Olajide, and R.A. Adenodi, Comparative Study of the Bisection and Newton Methods in solving for Zero and Extremes of a Single-Variable Function, *Journal of Nigerian Association of Mathematical Physics.* 21(1) 173-176 (2012).
3. N.D. Biswa. Numerical Solution of root Finding Problems:http://www.math.niu.edu/~dattab/math435/LectureNotes (2012)
4. J. Traub. Iterative methods for the solution of equations. *Mathematics.* (1982).
5. P. Miodrag., M. S. Petkovic., L. D. Petkovic. Complex interval arithmetic and its applications, *John Wiley & Sons.* Pg 284: (1998).
6. A.K. Singh, M. Kumar, and A. Srivastava, A New Fifth Order Derivative-Free Newton-type Method for Solving Nonlinear Equations, *Applied Mathematics & Information Sciences an International Journal,* 9(3) 1507-1513 (2015).
7. S.M. King., A. Rafiq, and Y.C. Kwun, A New Second-Order Iteration Method for Solving Nonlinear Equations, Special Issue in *Abstract and Applied Analysis.* (2013)
8. W. Gautschi, Numerical analysis: An Introduction, *Springer Science and Business Media, New York* (2011).
9. M.A. Noor, K.I. Noor, and M. Waseem, Fourth-order iterative methods for solving nonlinear equations, *International Journal of Applied Mathematics and Engineering Sciences,* 4(1) 43–52 (2010).
10. K.I. Noor, and M.A. Noor, Predictor-corrector Halley method for nonlinear equations, *Applied Mathematics and Computation,* 188(2)1587–1591 (2007).
11. S Akram, and Q.U. Ann, Newton Raphson Method, *International Journal of Scientific & Engineering Research.* 6: (2015).
12. R. Soram, S. Roy, S.R. Singh, M. Khomdram, S. Yaikhom, and S. Takhellambam, On the Rate of Convergence of Newton-Raphson Method. *The International Journal of Engineering and Science (IJES).* 2 (11) 5-12 (2013).
13. S.M. Kang, Improvements in Newton-Raphson Method for Non-linear Equations Using Modified Adomian Decomposition Method. *International Journal of Mathematical Analysis*. 9: (2015).
14. M. Kumar, A.K. Singh, and A. Srivastava, Various Newton-type iterative methods for solving nonlinear equations. *Journal of the Egyptian mathematical society.* 21(3) 334-339 (2013).
15. M.S.M. Bahgat, and M.A. Hafiz, Eighteenth Order Convergence for Solving Nonlinear Equations, *Department of Mathematics Faculty of Science El-Minia University EGYPT and Faculty of Science and Arts Najran University Najran, Saudi Arabia.* (2014).
16. L. Fang, L. Ni, and R. Chen, Three Modified Efficient Iterative Methods for Non-linear Equations, *Mathematical Computation,* 2(1) 6-12 (2013).
17. W. Nazeera, Generalized Newton Raphson's method free from second derivative, *Journal of Nonlinear Sci. Applied* 9: 2823–2831 (2016).