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Research Article

A Generalization of Lindley Distribution with Properties and Application

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Abstract: Lifetime data analysis is an important part of the Physical sciences, Engineering, and insurance sectors. The present study proposed a generalized form of Lindley distribution, which is a lifetime model, and developed its Statistical properties including moment generating function, generalized moment function, mean deviations, Bonferroni and Lorenz curve, reliability measures (survival function, hazard function, cumulative hazard function, reversed hazard function, mean residual life function), order statistics and Reyni entropy. An application of real-life data showed that the proposed model best fitted the data.

Keywords: Lifetime, Reliability, Survival, Hazard, Entropy, Order statistics.

1. INTRODUCTION

The lifetime data analysis is in the Physical sciences, and Engineering, and life insurance sectors. For this purpose, many distributions including exponential, Gamma, Rayleigh, Weibull, and Lindley, etc. have been developed. Recently introduced lifetime data distributions include two and three parameters Lindley, Quasi Lindley, Sujatha, Akash, Shanker, Aradhana, Suja, Amerandra, Ishita, Odoma, Rani, Deyvia, Pranav Sushila and Shukla distributions, which are all of mixture type distributions under mixture proportions with gamma and exponential distributions. The Lindley (1958) distribution with pdf and cdf respectively [1] given as:

$$f_L(\mathrm{y},\theta) = \frac{\theta^2}{\theta+1}(1+y)e^{-\theta y} \qquad (1.1)$$

$$y > 0;\ \theta > 0$$

$$F_L(\mathrm{y},\theta) = 1 - \frac{\theta y+\theta+1}{\theta+1}e^{-\theta y} \qquad (1.2)$$

is a mixture of $f_1(\mathrm{y})$ as an Exponential(θ) and $f_2(\mathrm{y})$ as a Gamma $(2,\theta) = \frac{\theta^2 y e^{-\theta y}}{\Gamma 2}$ with mixture proportions $P_1 = \frac{\theta}{\theta+1}$ and $P_2 = \frac{1}{\theta+1}$. Quasi Lindley distribution proposed by Shanker et al in 2016 , is a mixture of exponential (θ) and $f_2(\mathrm{y})$ as a Gamma $(2,\theta) = \frac{\theta^2 y e^{-\theta y}}{\Gamma 2}$ with mixture proportions $P_1 = \frac{\alpha}{\alpha+1}$ and $P_2 = \frac{1}{\alpha+1}$ [2] i.e.

$$f(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) \qquad (1.3)$$

Nadarajah et al. (2011) developed a generalized Lindley distribution and evaluated its properties[3]. Shanker et al. (2013) introduced two-parameter Lindley distribution, with pdf and cdf [4] given as:

$$f_{TPLD}(\mathrm{y},\theta,\alpha) = \frac{\theta}{\alpha\theta+1}(\alpha+y)e^{-\theta y}, \qquad (1.4)$$

$$y > 0;\ \theta > 0;\ \alpha\theta > -1,$$

$$F_{TPLD}(\mathrm{y},\theta,\alpha) = 1 - \left(1 + \frac{\theta y}{\alpha\theta+1}\right)e^{-\theta y}. \quad (1.5)$$

Three parameters Lindley distribution (Three PLD), introduced by Shanker et al. (2017), has the following pdf and cdf [5]:

$$f_{TPLD}(\mathrm{y},\theta,\alpha) = \frac{\theta^2}{\alpha\theta+k}(\alpha+ky)e^{-\theta y},$$

$$where\quad y > 0;\ \theta > 0;\ \alpha\theta > -1;\ k > 0 \quad (1.6)$$

$$F_{TPLD}(\mathrm{y},\theta,\alpha) = 1 - \left(1 + \frac{k\theta y}{\alpha\theta+k}\right)e^{-\theta y} \quad (1.7)$$

Shanker (2015)[6] introduced Akash distribution with pdf and cdf [6] respectively,

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$$f_{AK}(\mathrm{y},\theta)=\frac{\theta^3}{\theta^2+2}(1+y^2)e^{-\theta y},\ y>0;\ \theta>0, \quad (1.8)$$

$$F_{AK}(\mathrm{y},\theta)=1-\frac{\theta y(\theta y+1)}{\theta^2+2}e^{-\theta y} \quad (1.9)$$

Shukla (2018) proposed Pranav distribution pdf $f_P(\mathrm{y},\theta)$ and cdf $F_P(\mathrm{y},\theta)$, [7] Shukla (2018) introduced Ram Awadh distribution with pdf $f_{RA}(\mathrm{y},\theta)$ and cdf $F_{RA}(\mathrm{y},\theta)$, and gave its properties and application, [8] Shanker et al. (2017) developed Suja distribution with pdf $f_{SJ}(\mathrm{y},\theta)$ and cdf $F_{SJ}(\mathrm{y},\theta)$ respectively [8] given below:

$$f_P(\mathrm{y},\theta)=\frac{\theta^4}{\theta^4+6}(1+y^3)e^{-\theta y}:y>0,\theta>0 \quad (1.10)$$

$$F_P(\mathrm{y},\theta)=1-\left[1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{\theta^4+6}e^{-\theta y}\right] \quad (1.11)$$

$$f_{RA}(\mathrm{y},\theta)=\frac{\theta^6}{\theta^6+120}(1+y^5)e^{-\theta y};\ y>\ 0;\ \theta>0, \quad (1.12)$$

$$F_{AK}(\mathrm{y},\theta)=1-\frac{\theta y(\theta^4y^4+5\theta^3y^3+20\theta^2y^2+60\theta y+120)}{\theta^6+120}e^{-\theta y} \quad (1.13)$$

$$f_{SJ}(\mathrm{y},\theta)=\frac{\theta^5}{\theta^4+24}(1+y^4)e^{-\theta y};\ y>0;\ \theta>0 \quad (1.14)$$

$$F_{SJ}(\mathrm{y},\theta)=1-\left[1+\frac{(\theta^4y^4+4\theta^3y^3+12\theta^2y^2+24\theta y)}{\theta^4+6}e^{-\theta y}\right] \quad (1.15)$$

2. A GENERALIZATION OF LINDLEY DISTRIBUTION (AGLD)

The generalized function of mixture type lifetime data distributions named as a Generalization of Lindley distribution (AGLD), has the p.d.f. and c.d.f. defined as the following respectively:

$$f_{AGLD}(y;\ \theta,\alpha,k)=\frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{\theta^k\Gamma k+\alpha\Gamma(k+1)};\ \ \theta>0;y>0;\ \alpha\geq 0;\ k\geq 0 \quad (2.1)$$

$$F_{AGLD}(y;\ \theta,\alpha,k)=1-\frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)} \quad (2.2)$$

Where $\Gamma(k+1,\theta y)$ is an upper incomplete gamma function, AGLD is a mixture of $Gamma\ (1,\theta)$ i.e. Exponential (θ) and $Gamma\ (k+1,\theta)=\frac{\theta^{k+1}y^k e^{-\theta y}}{\Gamma(k+1)}$ with mixture proportions $P_1=\frac{\theta^k\Gamma k}{\theta^k\Gamma k+\alpha\Gamma(k+1)}$ and $P_2=\frac{\alpha\Gamma(k+1)}{\theta^k\Gamma k+\alpha\Gamma(k+1)}$ respectively. For the value of $k=\alpha=1$ it is Lindley distribution (1.1), $\alpha=0$ it reduced to $Exponential\ (\theta)$, for $k=2,\alpha=1$ it reduced to Akash distribution (1.8).

3. MATHEMATICAL AND STATISTICAL PROPERTIES

Moment generating function: The moment generating function of generalization of Lindley given as;

$$M_y(t)=\sum_{m=0}^{\infty}\frac{\{\theta^k\Gamma k\Gamma m+1+\alpha\Gamma(k+m+1)\}}{m!\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}}\left(\frac{t}{\theta}\right)^m \quad (3.1)$$

3.1 Moments about the Origin

The moment about the origin of generalized function for mixture distributions defined as;

$$\mu_r{}'=\frac{\{\theta^k\Gamma k\Gamma r+1+\alpha\Gamma(k+r+1)\}}{\theta^r\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}}\qquad r=1,2,3,\ldots \quad (3.2)$$

1st four moments about the origin are:

$$\mu_1{}'=\frac{\{\theta^k\Gamma k+\alpha\Gamma(k+2)\}}{\theta\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (3.3)$$

$$\mu_2{}'=\frac{\{2\theta^k\Gamma k+\alpha\Gamma(k+3)\}}{\theta^2\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (3.4)$$

$$\mu_3{}'=\frac{\{6\theta^k\Gamma k+\alpha\Gamma(k+4)\}}{\theta^3\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (3.5)$$

$$\mu_4{}'=\frac{\{24\theta^k\Gamma k+\alpha\Gamma(k+5)\}}{\theta^4\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (3.6)$$

$$Mean=\frac{\{\theta^k\Gamma k+\alpha\Gamma(k+2)\}}{\theta\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (3.7)$$

$$Var(y)=\frac{[\theta^{2k}(\Gamma k)^2+\alpha\theta^k\Gamma k\Gamma(k+3)+\alpha^2\Gamma(k+3)\Gamma(k+2)-\{\Gamma(k+2)\}^2]}{\theta^2\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^2} \quad (3.8)$$

$$CV=\frac{\sqrt{[\theta^{2k}(\Gamma k)^2+\alpha\theta^k\Gamma k\Gamma(k+3)+\alpha^2\Gamma(k+3)\Gamma(k+2)-\{\Gamma(k+2)\}^2]}}{\{\theta^k\Gamma k+\alpha\Gamma(k+2)\}}\times\times 100 \quad (3.9)$$

index of dispersion,

$$\gamma=\frac{\sigma^2}{\mu_1{}'}=\frac{\beta[\theta^{2k}(\Gamma k)^2+\alpha\theta^k\Gamma k\Gamma(k+3)+\alpha^2\Gamma(k+3)\Gamma(k+2)-\{\Gamma(k+2)\}^2]}{\theta\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}\{\theta^k\Gamma k+\alpha\Gamma(k+2)\}} \quad (3.10)$$

3.2. Mean Deviations

The mean deviation, usually taken from the mean or the median, is a measure of variation in a population. These are known as the mean deviation about the mean and the mean deviation about the median and respectively defined by;

$$\varphi_1(y) = \int_0^\infty |Y-\mu| f_{AGLD}(y)dy, \quad (3.11)$$

$\mu = E(y)$

$$\varphi_2(y) = \int_0^\infty |y-M| f_{AGLD}(y)dy \quad (3.12)$$

$M = Median(y)$ and $\varphi_1(y)$ and $\varphi_2(y)$ can be calculated as:

$$\varphi_1(y) = \int_0^\mu (\mu-y) f_{AGLD}(y)dy + \int_\mu^\infty (y-\mu) f_{AGLD}(y)dy$$

$$= \mu F(\mu) - \int_0^\mu y f_{AGLD}(y)dy - \mu[1-F(\mu)] + \int_\mu^\infty y f_{AGLD}(y)dy$$

$$= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty y f_{AGLD}(y)dy$$

$$= 2\mu F(\mu) + 2\int_0^\mu y f_{AGLD}(y)dy \quad (3.13)$$

The mean deviation about median defined as;

$$\varphi_2(y) = \int_0^\infty |y-M| f_{AGLD}(y)dy,$$

$$\varphi_2(y) = \int_0^M (M-y) f_{AGLD}(y)dy + \int_M^\infty (y-M) f_{AGLD}(y)dy,$$

$$= MF(M) - \int_0^M y f_{AGLD}(y)dy - M[1-F(M)] + \int_M^\infty y f_{AGLD}(y)dy,$$

$$= 2MF(M) - \int_0^M y f_{AGLD}(y)dy - M + \int_M^0 y f_{AGLD}(y)dy + \int_0^\infty y f_{AGLD}(y)dy,$$

$$= -\int_0^M y f_{AGLD}(y)dy + \int_M^0 y f_{AGLD}(y)dy + \mu,$$

$$= \mu - 2\int_0^M y f_{AGLD}(y)dy, \quad (3.14)$$

So by using pdf (2.1)

$$\int_0^\mu y f_{AGLD}(y)dy = \frac{\{\Gamma k(1-e^{-\theta\mu}-\theta\mu e^{-\theta\mu}) + \alpha\gamma(k+2,\theta\mu)\}}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}, \quad (3.15)$$

$$\int_0^M y f_{AGLD}(y)dy = \frac{\{\Gamma k(1-e^{-\theta M}-\theta M e^{-\theta M}) + \alpha\gamma(k+2,\theta M)\}}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}, \quad (3.16)$$

Where $\gamma(k+2,\theta\mu)$, $\gamma(k+2,\theta M)$ are lower incomplete gamma functions.

Put (3.15) in (3.13) and (3.16) into (3.14) have the following:

$$\varphi_1(y) = \frac{2\left\{\begin{matrix}(1-\mu)\alpha\gamma(k+1,\theta\mu) + \alpha\mu\Gamma(k+1) + \\ \Gamma k(1-e^{-\theta\mu}-\theta\mu e^{-\theta\mu}+\mu\theta^k) - \mu\theta^k e^{-\theta\mu}\end{matrix}\right\}}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}$$

(3.17)

$$\varphi_2(y) = \frac{\left[\begin{matrix}\theta^k\Gamma k + \alpha\Gamma(k+1) - \\ 2\{\Gamma k(1-e^{-\theta M}-\theta M e^{-\theta M}) + \alpha\gamma(k+2,\theta M)\}\end{matrix}\right]}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}.$$

(3.18)

4. BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves are introduced by Bonferroni(1930) [9]. Bonferroni and Gini indices have been utilized in economics, to study the variation in income and poverty, in other fields like reliability, vital statistics, insurance, and medicine. The Bonferroni and Lorenz curves are defined as:

$$B(p) = \frac{1}{p\mu}\int_0^q y f_{AGLD}(y)dy, \quad (4.1)$$

$$= \frac{1}{p\mu}\int_0^\infty y f_{AGLD}(y)dy - \frac{1}{p\mu}\int_q^\infty y f_{AGLD}(y)dy,$$

$$= \frac{1}{p} - \frac{1}{p\mu}\int_q^\infty y f_{AGLD}(y)dy.$$

By using (2.1) pdf of AGLD

$$\int_0^q y f_{AGLD}(y)dy = \frac{\{\Gamma k(1-e^{-\theta q}-\theta q e^{-\theta q}) + \alpha\gamma(k+2,\theta q)\}}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}, \quad (4.2)$$

Put (4.2) into (4.1).

$$B(p) = \frac{1}{p\mu}\frac{\{\Gamma k(1-e^{-\theta q}-\theta q e^{-\theta q}) + \alpha\gamma(k+2,\theta q)\}}{\theta\{\theta^k\Gamma k + \alpha\Gamma(k+1)\}}, \quad (4.3)$$

$$L(p) = \frac{1}{\mu}\int_0^q y f_{AGLD}(y)dy,$$

$$= \frac{1}{\mu}\int_0^{\infty} y f_{AGLD}(y)dy - \frac{1}{\mu}\int_q^{\infty} y f_{AGLD}(y)dy,$$

$$= 1 - \frac{1}{\mu}\int_q^{\infty} y f_{AGLD}(y)dy. \qquad (4.4)$$

By substituting (4.2) into (4.4),

$$L(p) = \frac{\{\Gamma k(1-e^{-\theta q}-\theta q e^{-\theta q})+\alpha\gamma(k+2,\theta q)\}}{\mu\theta\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \qquad (4.5)$$

Or both equivalent to $B(p) = \frac{1}{p\mu}\int_0^p F(y)^{-1}dy$ and $L(p) = \frac{1}{\mu}\int_0^p F(y)^{-1}dy$ where $q = F(p)^{-1}$

The Bonferroni indices is defined as:

$$B = 1 - \int_0^1 B(p)dp \qquad (4.6)$$

Put (4.3) into (4.6) has,

$$B = 1 - \frac{\{\Gamma k(1-e^{-\theta q}-\theta q e^{-\theta q})+\alpha\gamma(k+2,\theta q)\}}{\theta\mu\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \qquad (4.7)$$

The Gini indices defined as following:

$$G = 1 - 2\int_0^1 L(p)dp \qquad (4.8)$$

Put (4.5) into (4.8) determined as:

$$G = 1 - \frac{2\{\Gamma k(1-e^{-\theta q}-\theta q e^{-\theta q})+\alpha\gamma(k+2,\theta q)\}}{\mu\theta\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \qquad (4.9)$$

5. RELIABILITY MEASURES

There are different reliability measures namely Survival Function, Hazard Rate Function, Cumulative Hazard Function, and Reversed Cumulative Hazard Function given below;

5.1. Survival Function

Let Y be a continuous random variable with pdf $f_{AGLD}(y;\ \theta,\alpha,k)$ (2.1) and cdf $F_{AGLD}(y;\ \theta,\alpha,k)$ (2.2) of AGLD the Survival function obtains as:

$$S_{AGLD}(y;\ \theta,\alpha,k) = \frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)} \qquad (5.1)$$

5.2. Hazard Function

Let Y be a continuous random variable with p.d.f. $f_{AGLD}(y;\theta,\alpha,k)$ (2.1) and c.d.f. $F_{AGLD}(y;\theta,\alpha,k)$ (2.2) AGLD The hazard rate function known as the failure rate function defined as:

$$h_{AGLD}(y;\ \theta,\alpha,k) = \lim_{\Delta y\to 0}\frac{P(Y<y+\Delta y|Y>y)}{\Delta y} = \frac{f(y;\ \theta,\alpha,k)}{1-F(y;\ \theta,\alpha,k)} \qquad (5.2)$$

By using (2.1) and (2.2) find as:

$$h_{AGLD}(y;\ \theta,\alpha,k) == \frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}} \qquad (5.3)$$

Here note that:

$$h_{AGLD}(0;\ \theta,\alpha,\beta,k) = f_{AGLD}(0;\ \theta,\alpha,\beta,k)$$

5.3. Cumulative Hazard Function

Let Y be a continuous random variable with p.d.f. $f_{AGLD}(y;\ \theta,\alpha,k)$ and c.d.f. $F_{AGLD}(y;\ \theta,\alpha,k)$ of AGLD then the Cumulative hazard function defined as:

$$CH_{AGLD}(y;\ \theta,\alpha,k) = -\ln|F(y;\ \theta,\alpha,k)| \qquad (5.4)$$

By putting (2.2) we have,

$$CH_{AGLD}(y;\ \theta,\alpha,k) = -\ln\left|1 - \frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)}\right| \qquad (5.5)$$

5.4. Reversed Hazard Function

Let Y be a continuous random variable with p.d.f. $f_{AGLD}(y;\ \theta,\alpha,k)$, c.d.f. $F_{AGLD}(y;\ \theta,\alpha,k)$ of AGLD then the reversed hazard function defined as:

$$H_{AGLD}(y) = \frac{f(y;\alpha,\theta,k)}{F(y;\alpha,\theta,k)} \qquad (5.6)$$

By putting (2.1) and (2.2) into (5.8) have found

$$H_{AGLD}(y;\alpha,\theta) = \frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{[\theta^k\Gamma k+\alpha\Gamma(k+1)-\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}]} \qquad (5.7)$$

6. ODER STATISTICS (OS)

The density function $f_{(i,j)}(y)$ of "ith" order statistics (i=1, 2, …, n) from independent and identically distributed (i.i.d), random variable $y_1, y_2, \dots y_n$. The order statistics say as $y_{(1)}, y_{(2)}, \dots y_{(n)}$ the function of order statistics defined as:

$$f_{(i,j)}(y;\alpha,\theta) = \frac{f(y)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}[F(y)]^{i+j-1} \qquad (6.1)$$

$f_{(i,j)}(y;\alpha,\theta) = \frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{B(i,n-i+1)\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}\left[1-\frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)}\right]^{i+j-1}$, $B(i,n-i+1)$ is a beta function, Substituting $f_{AGLD}(y;\ \theta,\alpha,k)$ (2.1), $F_{AGLD}(y;\ \theta,\alpha,k)$ (2.2) into (6.1), the pdf of ith order statistics is:

$$f_{(i,j)}(y;\ \theta,\alpha,k) = \frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{B(i,n-i+1)\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}}\times \sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}\sum_{m=0}^{\infty}\binom{i+j-1}{m}\times \left[\frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)}\right]^m \quad (6.2)$$

The ith order statistics c.d.f is:

$$F_{(i,j)}(y;\ \theta,\alpha,k) = \sum_{j=0}^{n}\sum_{i=0}^{n-j}(-1)^i\binom{n}{j}\binom{n-j}{i}\left\{1-\frac{\{\theta^k e^{-\theta y}+\alpha\Gamma(k+1,\theta y)\}}{\theta^k\Gamma k+\alpha\Gamma(k+1)}\right\}^{j+i} \quad (6.3)$$

For maximum order, statistics put i=n, for minimum order statistics put i=1 in the equation.

7. RENYI ENTROPY MEASURE

A popular entropy measure is Renyi entropy (1961), and the entropy of a random variable Y is a measure of the variation of uncertainty [10]. Let Y is a continuous random variable having probability density function AGLD $f_{AGLD}(y;\ \theta,\alpha,k)$, then Renyi entropy is defined as;

$$T_{RE}(y) = \frac{1}{1-\delta}\log\{\int_0^\infty f(y)^\delta\, dy\} \quad (7.1)$$

Let $f_{AGLD}(y;\alpha,\theta,k)$ pdf of AGLD the Renyi entropy such that:

$$T_{RE}(y;\alpha,\theta,k) = \frac{1}{1-\delta}\log\{\int_0^\infty f_{AGLD}(y;\ \theta,\alpha,k)^\delta\, dy\}$$

$$= \frac{1}{1-\delta}\log\left\{\int_0^\infty\left[\frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}}\right]^\delta dy\right\}$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^\delta}\int_0^\infty(\Gamma k+\alpha y^k)^\delta e^{-\theta\delta y}dy\right]$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}(\Gamma k)^\delta}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^\delta}\int_0^\infty\left\{1+\frac{\alpha y^k}{\Gamma k}\right\}^\delta e^{-\theta\delta y}dy\right]$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}(\Gamma k)^\delta}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^\delta}\int_0^\infty\sum_{j=0}^{\infty}\binom{\delta}{j}(\frac{\alpha y^k}{\Gamma k})^j e^{-\theta\delta y}dy\right]$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}(\Gamma k)^\delta}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^\delta}\int_0^\infty\sum_{j=0}^{\infty}\binom{\delta}{j}(\frac{\alpha}{\Gamma k})^j y^{jk} e^{-\theta\delta y}dy\right]$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}(\Gamma k)^\delta}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^\delta}\sum_{j=0}^{\infty}\binom{\delta}{j}(\frac{\alpha}{\Gamma k})^j\left(\frac{1}{\theta\delta}\right)^{jk}\Gamma(jk+1)\right] \quad (7.2)$$

8. MAXIMUM LIKELIHOOD METHOD

Let $y_1, y_2, ... y_n$ be an independent random sample from $f_{AGLD}(y;\ \theta,\alpha,k)$, the Maximum Likelihood (ML) function defined as:

$$L(y;\ \theta,\alpha,k) = \prod_{i=0}^{n}\frac{\theta^{k+1}(\Gamma k+\alpha y^k)e^{-\theta y}}{\theta^k\Gamma k+\alpha\Gamma(k+1)} \quad (8.1)$$

$$lnL(y;\ \theta,\alpha,k) = n(k+1)ln\theta + \sum_{i=0}^{n}\ln(\Gamma k+\alpha {y_i}^k) - \theta\sum_{i=0}^{n}y_i - nln\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\theta} = \frac{n(k+1)}{\theta} - \sum_{i=0}^{n}y_i - \frac{n`k\,(\theta)^{k-1}}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}} \quad (8.2)$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\alpha} = \sum_{i=0}^{n}\frac{{y_i}^k}{(\Gamma k+\alpha {y_i}^k)} - \frac{n\Gamma(k+1)}{\theta^k\Gamma k+\alpha\Gamma(k+1)} \quad (8.3)$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial k} = nln\theta + \sum_{i=0}^{n}\frac{\frac{d}{dk}\Gamma k+\alpha {y_i}^k ln y_i}{(\Gamma k+\alpha {y_i}^k)} - \frac{n\left[(\theta)^k\{\ln(\theta)+\frac{d}{dk}\Gamma k\}+\alpha\frac{d}{dk}\Gamma(k+1)\right]}{(\Gamma k+\alpha {y_i}^k)} \quad (8.4)$$

By putting (8.2), (8.3), and (8.4) as $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\theta} = 0$; $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\alpha} = 0$, $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial k} = 0$. The eq. (8.2), (8.3), and (8.4) natural log-likelihood equations do not seem to be solved directly because they are not in closed form, so Fisher's scoring method can be applied to solve these equations. We have;

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\theta^2} = -\frac{n(k+1)}{\theta^2} + \frac{nk\,\theta^{k-2}\{\alpha(k-1)\Gamma(k+1)\,\theta^k - \Gamma k\}}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^2} \quad (8.5)$$

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\alpha^2} = \frac{{y_i}^{2k}}{(\Gamma k+\alpha {y_i}^k)^2} + n\left\{\frac{\Gamma(k+1)}{\theta^k\Gamma k+\alpha\Gamma(k+1)}\right\}^2 \quad (8.6)$$

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k^2} = -\frac{n\left[\{(\theta\alpha)^k+\Gamma(k+1)\}\left\{\frac{d}{dk}\Gamma(k+1)\right\}-\{\Gamma(k+1)\}^2\right]}{\{(\theta\alpha)^k+\Gamma(k+1)\}^2} + \sum_{i=0}^{n}\frac{(\Gamma k+\alpha {y_i}^k)\frac{d^2}{dk^2}\Gamma k - \left(\frac{d}{dk}\Gamma k+\alpha {y_i}^k ln y_i\right)^2}{(\Gamma k+\alpha {y_i}^k)^2} \quad (8.7)$$

$\frac{d\Gamma(k+1)}{dk}$ *and* $\frac{d\Gamma(k)}{dk}$ *is digmma functions*

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\theta\partial\alpha} = \frac{n`k\,(\theta)^{k-1}\Gamma(k+1)}{\{\theta^k\Gamma k+\alpha\Gamma(k+1)\}^2} \quad (8.8)$$

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k\partial\theta} = \frac{n}{\theta} - \frac{n\,(\theta)^{k-1}\left[1-k\left\{\ln(\theta)+\frac{d}{dk}\Gamma k\right\}\right]}{(\Gamma k+\alpha {y_i}^k)} \quad (8.9)$$

$\left\{\frac{d}{dk}\Gamma(k+1)\right\}$ *is digmma function*

$$\frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k\partial\alpha} = \sum_{i=0}^{n}\frac{\left(\Gamma k ln y_i - \frac{d}{dk}\Gamma k\right){y_i}^k}{(\Gamma k+\alpha {y_i}^k)^2} - \frac{n\left[\Gamma k\frac{d}{dk}\Gamma(k+1) - (\theta)^k {y_i}^k\left\{\ln(\theta)+\frac{d}{dk}\Gamma k\right\}\right]}{(\Gamma k+\alpha {y_i}^k)^2} \quad (8.10)$$

The iterative solution of the equations (8.5) to (8.10) using the matrix given following will be the MLEs $\hat{\theta}$ $\hat{\alpha}, and\ \hat{k}$ of parameters θ α and k of AGLD.

$$\begin{bmatrix} \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\theta^2} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\alpha\partial\theta} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k\partial\theta} \\ \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\theta\partial\alpha} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\alpha^2} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k\partial\alpha} \\ \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\theta\partial k} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial\alpha\partial k} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{k}=k_0}}$$

$$\begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{k}-k_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL(y;\theta,\alpha,k)}{\partial\theta} \\ \frac{\partial lnL(y;\theta,\alpha,k)}{\partial\alpha} \\ \frac{\partial lnL(y;\theta,\alpha,k)}{\partial k} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{k}=k_0}}$$

Here θ_0, α_0 and k_0 initial values of parameters of θ α β and k of AGLD.

9. APPLICATION

The data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994) and are given as 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381 [11].

To compare lifetime distributions, Kolmogorov-Smirnov Statistics (KSS), Pearson Chi-square and Anderson Darling statistics for the above data set have been computed their statistics defined respectively.

Kolmogorov-Smirnov statistics:

$$K-S = Sup_{(y)}|F_n(y)-F_0(y)|$$

$F_n(y)$= empirical distribution, n= sample size

Person Chi-square statistics:

$\chi^2 = \sum_{i=1}^{n}\frac{(O_i-E_i)^2}{E_i}$; O_i=Observed frequency, E_i=Expected frequency

Anderson Darling statistics:

$$A^2 = -n - S$$

Where $S = \sum_{i=1}^{n}(2i-1)/n[lnF(y_i)+\ln\{1-F(y_{n+1-i})\}]$

F is the cumulative distribution function of the specified distribution and y_i are ordered data.

Table 1. The MLE's, K-S statistics, Chi-square, and Anderson-Darling statistics values

Distributions	MLE's			K-S statistics	Chi-square statistics	Anderson-Darling
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$			
AGLD	0.61434	1.0182	17.93	0.134906	7.45161	0.438565
Ram Awadh	0.19473			0.197684	22.9355	2.00445
Suja	1.65272			0.223211	22.4194	2.54007
Pranav	0.12978			0.25346	26.5484	3.24745
Akash	0.09706			0.294639	34.8065	4.19329
Three PLD	0.06489	0.0210	2.00	0.358665	36.3548	5.69565
TPLD	0.06488	0.010		0.358653	36.3548	5.69540
Lindley	0.06298			0.365453	35.8387	5.87165
Exponential	0.03245			0.458623	60.0968	8.52724

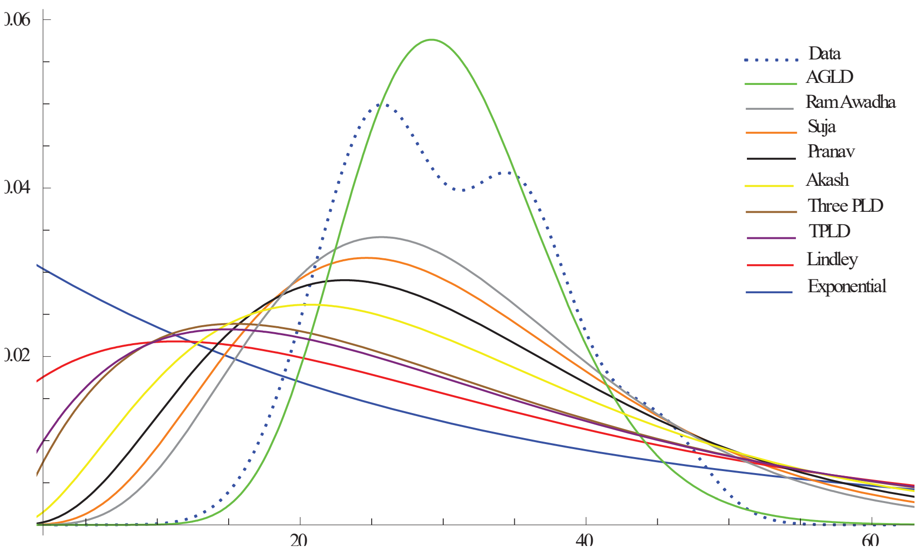

Fig 1. The goodness of fit of distributions

10. CONCLUSIONS

A generalization of Lindley distribution (AGLD) is a lifetime data analysis distribution with three parameters θ α and k, i.e. a mixture of Gamma $(1,\theta)$, Exponential (θ) and Gamma $(k+1,\theta)=\frac{\theta^{k+1}y^{k}e^{-\theta y}}{\Gamma(k+1)}$ with mixture proportions: $P_1=\frac{\theta^{k}\Gamma k}{\theta^{k}\Gamma k+\alpha\Gamma(k+1)}$ and $P_2=\frac{\alpha\Gamma(k+1)}{\theta^{k}\Gamma k+\alpha\Gamma(k+1)}$ respectively. For the value of $k=\alpha=1$ it is Lindley distribution, $\alpha=0$ it reduced to Exponential (θ), for $k=2,\alpha=1$ it reduced to Akash distribution. Moreover, mathematical and statistical properties including moment generating function, r^{th} moments function, mean deviations, Bonferroni and Lorenz curve,reliability measures (survival function, hazard function, cumulative hazard function, reversed hazard function, mean residual life function), order statistics, and Reyni entropy have been described. Also, the parameters of the distribution using the maximum likelihood estimation method are computed on basis of real-life data. The goodness of fit of AGLD is compared with other lifetime distributions using real-life data set. The values of K-S statistics (Kolmogorov-Smirnov statistics), Pearson chi-square, and Anderson Darling statistics are given in according to values of the goodness of fit statistics, AGLD best fitted the data than other distributions.

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