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Research Article

Further Results on Edge Product Cordial Labeling

Mohamed Aboshady[1], Reda Elbarkouky[2], Eliwa Roshdy[3], and Mohamed Abdel-Azim Seoud[4]

[1]Department of Basic Science, Faculty of Engineering, The British University in Egypt, Cairo, Egypt
[2]Department of Engineering Physics and Mathematics, Faculty of Engineering, Ain Shams University, Cairo, Egypt
[3]Department of Basic & Applied Sciences, Arab Academy for Science, Technology and Maritime Transport, Cairo, Egypt
[4]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt

Abstract: In this paper, we present some properties of Edge Product Cordial graphs and make a survey on all graphs of order at most 6 to find out whether they are Edge Product Cordial or not. Finally, we study some families of graphs to be Edge Product Cordial or not: $K_{2,n}^{(m)}$, $SF(n)$, $T_n \odot K_1$, $P_n \odot \overline{K_m}$, $C_n \odot \overline{K_m}$, $P_n \wedge C_m$ and the graph obtained from C_m by attaching a pendant path P_n to every vertex of C_m.

Keywords: Edge Product Cordial graph/labeling, Paths P_n, Cycles C_n.

1. INTRODUCTION

Graph labeling which often helps in increasing a group of mathematical models for a wide area of applications have also discovered applications in different coding theory problems, such as the designing of good radar-type codes, synch-set codes and convolutional codes with optimal autocorrelation properties. It is also used in the field of cellular radio systems, routing techniques and many other applications, see [1,2].

One of the graph labeling techniques is the edge product cordial labeling. The concept of edge product cordial (EPC) labeling was first presented by Vaidya & Barasara [1,3] as edge similar to product cordial labeling. An EPC labeling of a graph G is an edge labeling function $f: E(G) \rightarrow \{0,1\}$ which results in a vertex labeling function $f^*: V(G) \rightarrow \{0,1\}$ such that $f^*(u) = \prod \{f(uv) \mid uv \epsilon E(G)\}$ provided that the number of edges with the label 0 and the number of edges with the label 1 is equal or vary by 1 and the number of vertices with the label 0 and the number of vertices with the label 1 are equal or vary by 1. A graph with an EPC labeling is said to be an EPC graph.

Vaidya and Barasara [3-8] studied the EPC labeling for some graphs such as cycles, trees, crowns, helms, gears, owers, webs, shells, triangular snakes, double triangular snakes, quadrilateral snakes, double quadrilateral snakes and some combinations of graphs. Vaidya and Barasara [8] studied the product and EPC labeling of additional families of graphs.

Vaidya and Barasara had also proved that some graphs are not EPC for some cases such as: C_n for n even; $K_{m,n}$ for $m, n \geq 2$; K_n for $n \geq 4$; the one point union of t copies of C_n for t odd and n even; wheels; tadpoles $C_n@P_m$ for $m+n$ odd and $m<n$; shells S_n for even n; for n even double triangular snake DT_n, quadrilateral snake Q_n and double quadrilateral snake DQ_n; C_n^2 for $n>3$; double fans; P_n^2 for even n; $D_2(C_n)$;

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*Corresponding Author: Mohamed Abdel-Azim Seoud < M.A.Seoud@hotmail.com>

$M(C_n)$; $D_2(P_n)$; $T(C_n)$; $S'(C_n)$; $S'(P_n)$ for odd n; the tensor product of C_n and C_m if m or n odd; $P_m \times P_n$ and $C_m \times C_n$; and $P_n[P_2]$ and $C_n[P_2]$.

In [9] Prajapati and Shah gave an EPC labeling for some families of graphs. Prajapati and Patel in [10,11] provided some more results on EPC graphs.

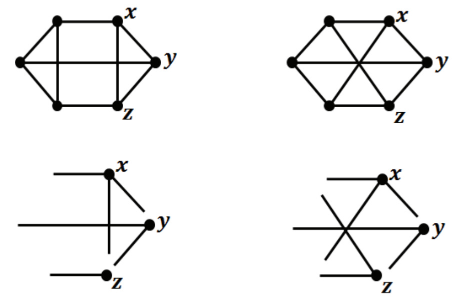
In spite of all these results on many families of graphs, in this paper we give some further general results on graphs satisfying the EPC labeling. We present some properties of EPC graphs, give an upper bound for the number of edges of any graph of a given order to be EPC, make an algorithm to check out any graph if it is EPC or not and study some families of graphs whether they satisfy the EPC labeling condition or not and present an EPC labeling to those that satisfy: $K_{2,n}^{(m)}$, $T_n \odot K_1$, $P_n \odot \overline{K_m}$, $C_n \odot \overline{K_m}$, $P_n \wedge C_m$ and the graph obtained from C_m by attaching a pendent path P_n to every vertex of C_m.

2. MAIN RESULTS

Definition 2.1 [3]: For a graph G, define the edge labeling function as $f: E(G) \to \{0,1\}$ and the induced vertex labeling function $f^*: V(G) \to \{0,1\}$ is given as if $e_1, e_2, \dots, e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1)f(e_2)\dots f(e_n)$. Let $v_f(i)$ be the number of vertices of G having label i under f^* and $e_f(i)$ be the number of edges of G having label i under f for $i = 0,1$. If $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$, then f is called EPC labeling of the graph G. If a graph G admits EPC labeling, then it is called EPC.

Theorem 2.2: For any regular graph $G(p,q)$ with degree $r > \left\lfloor \frac{p}{2} \right\rfloor - 1$. If $r > \frac{2\left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right)}{2\left\lfloor \frac{p}{2} \right\rfloor}$, then G is not EPC graph.

Proof: We will begin the proof by illustrating the following two regular graphs as an example. It's clear that the graph in Figure 1(a) requires less edges to be labeled with edge label "1" than the graph in Figure 1(b) so as to label half the vertices of each graph with vertex label "1".



(a) **(b)**

Fig. 1. Two regular graphs with $r = 3$, $p = 6$ and $q = 9$.

Then generalizing for any regular graph $G(p,q)$ with degree r, $r > \left\lfloor \frac{p}{2} \right\rfloor - 1$, to be EPC graph, the maximum number of edges allowed to be labeled with the edge label 1 is $\left\lceil \frac{q}{2} \right\rceil$, and the minimum number of vertices allowed to be labeled with the vertex label 1 is $\left\lfloor \frac{p}{2} \right\rfloor$. Since for any vertex to be labeled with a vertex label 1, all the incident edges to this vertex must be labeled with edge label 1 according to the definition of the labeling; assuming the minimum number of edges to be taken as shown in the example; then the following inequality must hold:

$$r + (r-1) + (r-2) + \cdots + \left(r - \left\lfloor \frac{p}{2} \right\rfloor + 1\right) \le \left\lceil \frac{q}{2} \right\rceil.$$

i.e. $$\left\lfloor \frac{p}{2} \right\rfloor r - \left(1 + 2 + \cdots + \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right)\right) \le \left\lceil \frac{q}{2} \right\rceil$$

$$\Leftrightarrow \left\lfloor \frac{p}{2} \right\rfloor r - \frac{\left\lfloor \frac{p}{2} \right\rfloor \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right)}{2} \le \left\lceil \frac{q}{2} \right\rceil.$$

Which leads to $r \le \frac{2\left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right)}{2\left\lfloor \frac{p}{2} \right\rfloor}$ which completes the proof by using contraposition.

Example 2.3: The graph $G(p = 8, q = 20)$ shown in Figure 2 is regular with $r = 5$. We see that $r + 1 = 6 > \left\lfloor \frac{p}{2} \right\rfloor = 4$, but $r = 5 > \frac{2\left\lceil \frac{q}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right)}{2\left\lfloor \frac{p}{2} \right\rfloor} = \frac{2(10) + 4(4-1)}{2(4)} = 4$. So, G is not an EPC graph.

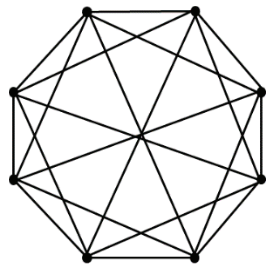


Fig. 2. A regular graph with r = 5, p = 8 and q = 20 is not EPC

Corollary 2.4: The upper bound for the number of edges of any regular graph $G(p,q)$ with degree r to be EPC graph is

$$q \le \frac{p(p-2)}{4} + 1, \quad \text{if } p \text{ is even}, p \ge 4 \text{ and}$$

$$q \le \frac{p(p^2-4p+7)}{4(p-2)} \qquad \text{if } p \text{ is odd}, p \ge 3.$$

Proof: Straightforward from the last theorem we will have two cases:

Case I: if p is even

$$\Rightarrow \left\lfloor \frac{p}{2} \right\rfloor = \frac{p}{2} \Rightarrow r = \frac{2q}{p} \le \frac{2\left\lceil \frac{q}{2} \right\rceil + \frac{p}{2}\left(\frac{p-2}{2}\right)}{2\left(\frac{p}{2}\right)}$$

$$\Rightarrow \left\lceil \frac{q}{2} \right\rceil \le \frac{p(p-2)}{8} \Rightarrow \begin{cases} q \le \frac{p(p-2)}{4} + 1, q \text{ is odd} \\ q \le \frac{p(p-2)}{4}, \quad q \text{ is even} \end{cases}.$$

Case II: if p is odd, then $\left\lfloor \frac{p}{2} \right\rfloor = \frac{p-1}{2}$

if q is even, then $\left\lceil \frac{q}{2} \right\rceil = \frac{q}{2} \Rightarrow r = \frac{2q}{p} \le \frac{q + \left(\frac{p-1}{2}\right)\left(\frac{p-3}{2}\right)}{p-1} \Rightarrow q \le \frac{p(p-1)(p-3)}{4(p-2)} = \frac{p(p^2-4p+3)}{4(p-2)}$,

if q is odd, then $\left\lceil \frac{q}{2} \right\rceil = \frac{q+1}{2} \Rightarrow r = \frac{2q}{p} \le \frac{q+1+(\frac{p-1}{2})(\frac{p-3}{2})}{p-1} \Rightarrow q \le \frac{p(p^2-4p+7)}{4(p-2)}$.

Which gives an upper bound for the number of edges of any regular graph $G(p,q)$ to be EPC graph.

Theorem 2.5: The upper bound for the number of edges of any graph G with order p to be EPC graph is $\left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) + 1$.

Proof: For any graph G with p vertices to be EPC graph, the minimum number of vertices labeled with the vertex label "1" is $\left\lfloor \frac{p}{2} \right\rfloor$. Since the maximum number of edges joining the remaining $\left\lceil \frac{p}{2} \right\rceil$-vertices is $\frac{\left\lceil \frac{p}{2} \right\rceil\left(\left\lceil \frac{p}{2} \right\rceil - 1\right)}{2}$ which will all be labeled with the edge label "0", so the number of edges labeled with the edge label "1" incident from the $\left\lfloor \frac{p}{2} \right\rfloor$-vertices will be at most $\frac{\left\lceil \frac{p}{2} \right\rceil\left(\left\lceil \frac{p}{2} \right\rceil - 1\right)}{2} + 1$. Adding the two sets of edges will lead to the upper bound which is $\left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) + 1$.

Corollary 2.6: For a connected graph $G(p,q)$. If $q \le \left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) + 1$ and there exist a complete subgraph H of G such that $|H| = \left\lceil \frac{p}{2} \right\rceil$, then G has EPC labeling.

Proof: Since the number of edges of the complete subgraph H is $\frac{\left\lceil \frac{p}{2} \right\rceil\left(\left\lceil \frac{p}{2} \right\rceil - 1\right)}{2}$ and $q \le \left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) + 1$. Then, the maximum number of edges of the graph $G - H$ such that G can be EPC graph is $\frac{\left\lceil \frac{p}{2} \right\rceil\left(\left\lceil \frac{p}{2} \right\rceil - 1\right)}{2} + 1$, then we will have 2 cases:

Case I: If $\left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) - 1 \le q \le \left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) + 1$, we will label all the edges of the graph $G - H$ with the label 1 and all the edges of the graph H with the label 0 leading to an EPC labeling of G.

Case II: If $q < \left\lceil \frac{p}{2} \right\rceil \left(\left\lceil \frac{p}{2} \right\rceil - 1\right) - 1$, we will label all the edges of the graph $G - H$ with the label 1 and complete the $\left\lceil \frac{q}{2} \right\rceil$-edges labeled with the label 1 from the subgraph H preserving a path with order $\left\lceil \frac{p}{2} \right\rceil$ (with the label 0 to all of its edges) passing through all the vertices of the subgraph H. And this will lead to an EPC labeling of G.

Example 2.7: The graph $G(p = 13, q = 30)$ shown in Figure 3 is connected with $K_7 \subseteq G$ and has an EPC labeling.

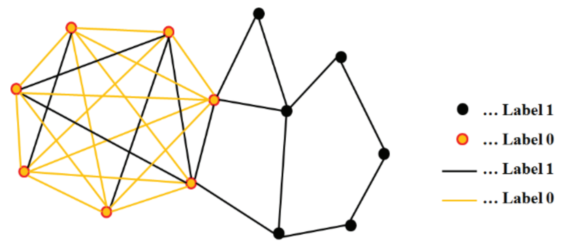


Fig. 3. An EPC graph G with $p = 13$ and $q = 30$ and $K_7 \subseteq G$

Algorithm 2.8: The following algorithm can be used to check out any graph $G(p, q)$ if it is EPC graph or not and give the vertex and edge labels for it if it is so as follows:

1. Enter p and q,
2. Enter the degree of each vertex,
3. Enter the edges incident to each vertex,
4. If q is even, then
   Label $\frac{q}{2}$ edges with the edge label "1", and the other edges with the edge label "0",
5. Else if q is odd, then
   Label $\frac{q+1}{2}$ edges with the edge label "1", and the other edges with the edge label "0",
6. Find the label of each vertex according to the EPC condition,
7. If the vertex labels satisfy the EPC labeling, then
   Display the edge and vertex labels. Stop.
8. If the vertex labels don't satisfy the EPC labeling, then
   Change the arrangement of the edge labels,
9. Go to step 6. Stop.

Theorem 2.9: All graphs $G(p \leq 6, q)$ are EPC graphs except the following graphs:

1. $G(p = 4, q \geq 4)$.
2. $G(p = 5, q \geq 8)$.
3. $G(p = 6, q \geq 8)$.
4. Among the graphs $G(p \leq 5, q \leq 7)$ the two graphs in Figure 4.

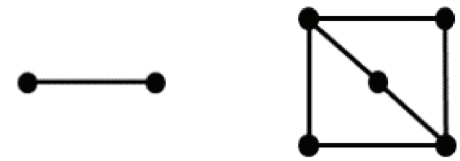


Fig. 4. Two graphs that are not EPC

5. Among the graphs $G(p = 6, q \leq 7)$ the graphs in Figure 5.

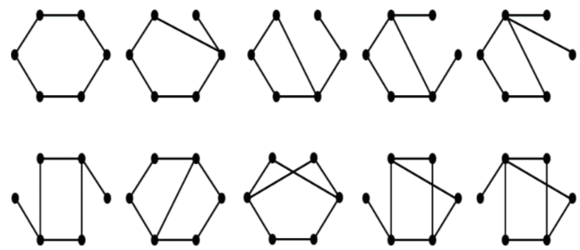


Fig. 5. Graphs $G(p = 6, q \leq 7)$ that are not EPC

Proof:

1. For $p = 4 \Longrightarrow \left\lceil \frac{p}{2} \right\rceil = 2$, then from the previous theorem: $q \leq 2(1) + 1 = 3$,
2. For $p = 5 \Longrightarrow \left\lceil \frac{p}{2} \right\rceil = 3$, then from the previous theorem: $q \leq 3(2) + 1 = 7$,
3. For $p = 6 \Longrightarrow \left\lceil \frac{p}{2} \right\rceil = 3$, then from the previous theorem: $q \leq 3(2) + 1 = 7$,
4. The first graph has only one edge, so the two vertices will have the same label. In the second graph, the distance between the two vertices with the smallest degrees is two, so we need at least four edges labeled with the edge label "1" to label two vertices with the vertex label "1".
5. Using the previous algorithm, these graphs are found to be not EPC graphs.

An EPC labeling of all graphs of order less than or equal 6 are shown in Figure 6.

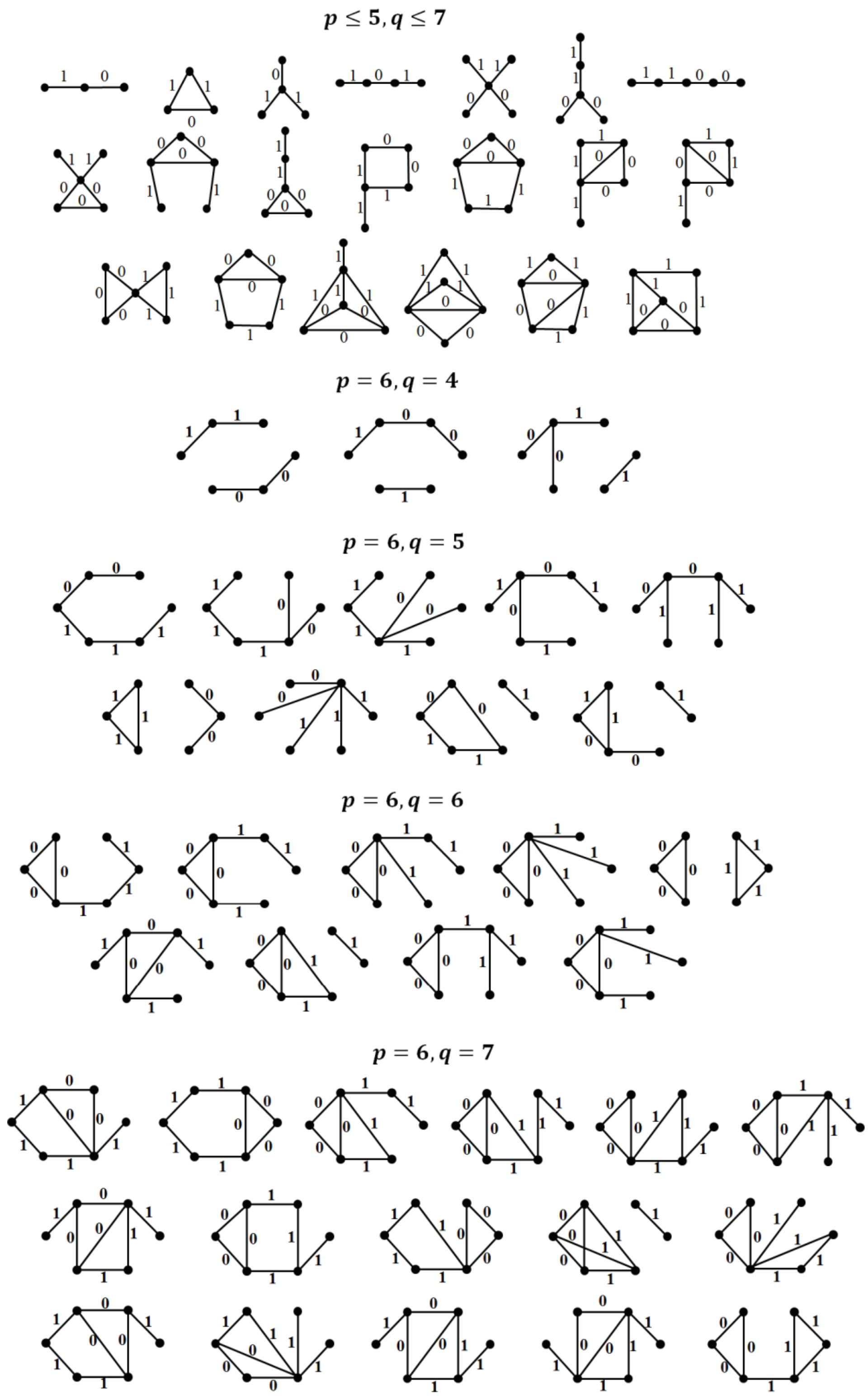


Fig. 6. An EPC labeling of all graphs of order at most 6

3. STUDY OF EPC LABELING FOR SOME FAMILIES OF GRAPHS

Theorem 3.1: The graph $K_{2,n}^{(m)}$; one point union of m copies of $K_{2,n}$; is an EPC if m is even.

Proof: This graph has $2nm$ edges and $nm+m+1$ vertices. Let $\{v_0; v_1, v_1^1, v_2^1, \ldots, v_n^1; v_2, v_1^2, v_2^2, \ldots, v_n^2; \ldots; v_m, v_1^m, \ldots, v_n^m\}$ be the set of vertices. If m is even, then we define the labeling function $f: E(K_{2,n}^{(m)}) \to \{0,1\}$ as follows:

$$f(v_0v_i^j) = f(v_jv_i^j) = 1; \quad 1 \le i \le n, 1 \le j \le \frac{m}{2}$$

$$f(v_0v_i^j) = f(v_jv_i^j) = 0; \quad 1 \le i \le n, \frac{m}{2}+1 \le j \le m$$

The above labeling will induce vertex labeling as follows:

$$f(v_0) = 0$$

$$f(v_j) = f(v_i^j) = 1, \quad 1 \le i \le n, 1 \le j \le \frac{m}{2}$$

$$f(v_j) = f(v_i^j) = 0, 1 \le i \le n, \frac{m}{2}+1 \le j \le m$$

From the above edge and vertex labelings we will have $e_f(0) = e_f(1) = nm$, and $v_f(0) = v_f(1) + 1 = \frac{(n+1)m}{2} + 1$. Therefore, for even m, $K_{2,n}^{(m)}$ is an EPC graph.

Example 3.2: $K_{2,3}^{(4)}$ is an EPC graph as shown in Fig. 7.

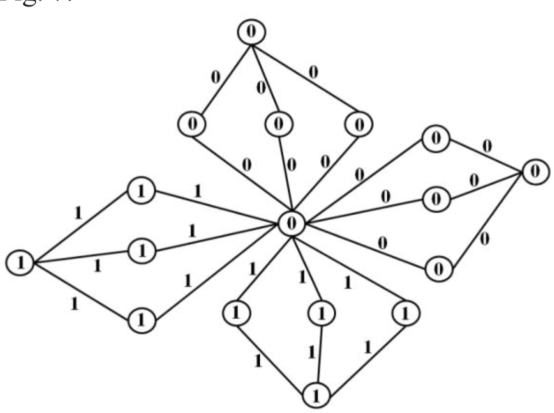


Fig. 7. An EPC labeling of $K_{2,3}^{(4)}$

Definition 3.3 [1]: The Sun Flower $SF(n)$ was characterized by Lee and Seah as the graph obtained from the cycle C_n with vertices $\{v_1, v_2, \ldots, v_n\}$ and the new vertices $\{u_1, u_2, \ldots, u_n\}$ such that $u_i, i = 1,2,\ldots,n-1$, is connected to v_i and v_{i+1}, and u_n is connected to v_n and v_1.

Theorem 3.4: The Sunflower $SF(n)$ is not an EPC graph.

Proof: This graph has n C_3-components with $2n$ vertices and $3n$ edges. Let $\{v_1, v_2, \ldots, v_n; u_1, u_2, \ldots, u_n\}$ be the set of vertices. Then, to satisfy the edge condition for the EPC graph, it is required to label at most $\left\lceil \frac{3n}{2} \right\rceil$-edges with the label 1 which will lead to at most $(n-1)$-vertex labeled with the label 1. Therefore, $|v_f(0) - v_f(1)| \ge 2$ which violates the vertex condition for the EPC graphs which completes the proof.

Definition 3.5: Harary [12] has defined the corona $G_1 \odot G_2$ of two graphs G_1 and G_2 as the graph obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2, where the i^{th} vertex of G_1 is joined to every vertex in the i^{th} copy of G_2.

Definition 3.6 [1]: The triangular snake T_n is the graph which is obtained from a path P_n with vertices $\{v_1, v_2, \ldots, v_n\}$ by joining the vertices v_i and v_{i+1} to a new vertex u_i for $i = 1,2,\ldots,n-1$.

Theorem 3.7: The graph $T_n \odot K_1$ is an EPC graph.

Proof: This graph has $|V(T_n \odot K_1)| = 4n-2$ vertices and $|E(T_n \odot K_1)| = 5n-4$ edges. Let $\{v_1, v_2, \ldots, v_n; u_1, u_2, \ldots, u_{n-1}; a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_{n-1}\}$ be the set of vertices as shown in Figure 8.

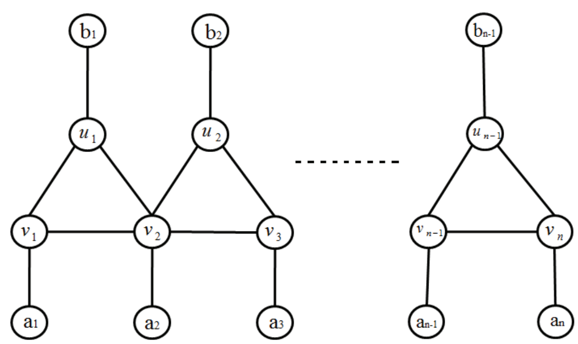


Fig. 8. The graph $T_n \odot K_1$

We define the labeling function $f : E(T_n \odot K_1) \rightarrow \{0,1\}$ as follows:

$$f(v_i a_i) = f(u_i b_{i-1}) = 1; \quad 1 \leq i \leq n$$
$$f(v_i v_{i+1}) = f(u_i v_{i+1}) = 0; \quad 1 \leq i \leq n-1$$
$$f(v_i v_i) = i + 1 \ (\mathrm{mod}\ 2); \quad 1 \leq i \leq n-1$$

The above labeling will induce vertex labeling as follows:

$$f(u_i) = f(v_i) = f(v_n) = 0, \qquad 1 \leq i \leq n-1$$
$$f(b_i) = f(a_i) = f(a_n) = 1, \qquad 1 \leq i \leq n-1$$

From the above edge and vertex labeling, we will have $v_f(0) = v_f(1) = 2n - 1$. Also $e_f(0) = \left\lfloor \frac{5n-4}{2} \right\rfloor$ and $e_f(1) = \left\lfloor \frac{5n-3}{2} \right\rfloor$ which means that $0 \leq |e_f(0) - e_f(1)| \leq 1$. Therefore, $T_n \odot K_1$ is an EPC graph.

Example 3.8: $T_4 \odot K_1$ and $T_5 \odot K_1$ are EPC graphs as shown in Figure 9 respectively.

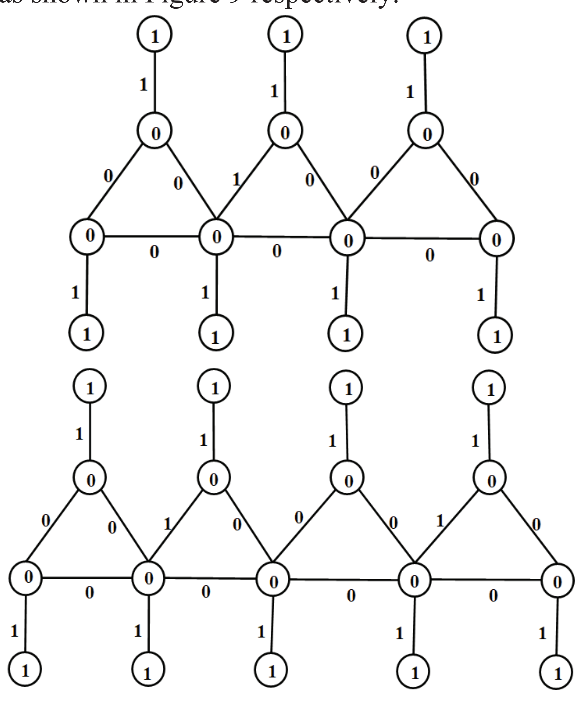


Fig. 9. An EPC labelings of the graphs $T_4 \odot K_1$ and $T_5 \odot K_1$ respectively

Theorem 3.9: The graph $P_n \odot \overline{K_m}$ is an EPC graph.

Proof: This graph has $|V(P_n \odot \overline{K_m})| = nm + n$ vertices and $|E(P_n \odot \overline{K_m})| = nm + n - 1$ edges. Let $\{v_1, v_1^1, v_1^2, \dots, v_1^m; v_2, v_2^1, v_2^2, \dots, v_2^m; \dots; v_n, v_n^1, v_n^2, \dots, v_n^m\}$ be the set of vertices as shown in Figure 10.

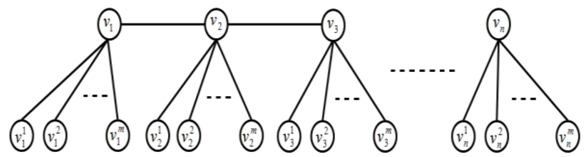


Fig. 10. The graph $P_n \odot \overline{K_m}$

Let $a = \left\lceil \frac{nm+n-1}{2} \right\rceil$ and apply the division algorithm to get $a = mk + d, 0 \leq d < m$, where $k = \left\lfloor \frac{a}{m} \right\rfloor$. Then define the labeling function $f : E(P_n \odot \overline{K_m}) \rightarrow \{0,1\}$ as follows:

$$f(v_i v_{i+1}) = 0, 1 \leq i \leq n-1$$
$$f\left(v_i v_i^j\right) = 0; \quad k+2 \leq i \leq n, 1 \leq j \leq m$$
$$f\left(v_{k+1} v_{k+1}^j\right) = 0; \ d+1 \leq j \leq m$$

and the remaining edges will be labeled “1".
Which will induce the vertex labeling as follows

$$f(v_i) = 0, \qquad 1 \leq i \leq n$$
$$f\left(v_i^j\right) = 0; \quad k+2 \leq i \leq n, 1 \leq j \leq m$$
$$f(v_{k+1}^j) = 0; \ d+1 \leq j \leq m$$

and the remaining vertices will be labeled “1".

From the above edge and vertex labelings, we will have $0 \leq |v_f(0) - v_f(1)| \leq 1$, and $0 \leq |e_f(1) - e_f(0)| \leq 1$. Therefore, $P_n \odot \overline{K_m}$ is an EPC graph.

Example 3.10: $P_3 \odot \overline{K_5}$ with $d = 4$ and $P_8 \odot \overline{K_2}$ with $d = 0$ are EPC graphs as shown in Figure 11 respectively.

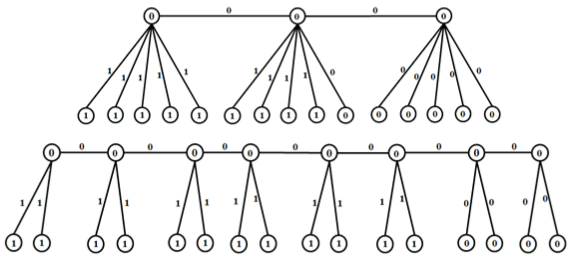


Fig. 11. An EPC labeling of the graphs $P_3 \odot \overline{K_5}$ and $P_8 \odot \overline{K_2}$ respectively

Corollary 3.11: The graph $C_n \odot \overline{K_m}$ is an EPC graph.

Proof: This graph has $|V(C_n \odot \overline{K_m})| = nm + n$ vertices and $|E(C_n \odot \overline{K_m})| = nm + n$ edges. Let $\{v_1, v_1^1, v_1^2, \dots, v_1^m; v_2, v_2^1, v_2^2, \dots, v_2^m; \dots; v_n, v_n^1, v_n^2, \dots, v_n^m\}$ be the set of vertices as shown in Figure 12.

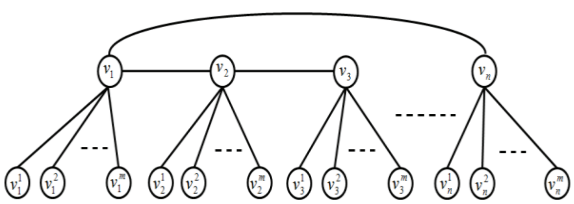


Fig. 12. The graph $C_n \odot \overline{K_m}$

We define the labeling function $f : E(C_n \odot \overline{K_m}) \to \{0,1\}$ as in the previous theorem and:

$$f(v_1 v_n) = 0$$

Which will induce the same vertex labeling as in the previous theorem.

Definition 3.12 [12]: The conjunction $G = G_1 \wedge G_2$ has the vertex set $V = V(G_1) \times V(G_2)$ and the two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever u_1, v_1 are adjacent and u_2, v_2 are adjacent.

Theorem 3.13: The graph $P_n \wedge C_m$ is an EPC graph if m is even.

Proof: This graph has $|V(P_n \wedge C_m)| = nm$ vertices and $|E(P_n \wedge C_m)| = 2m(n-1)$ edges. Let $\{v_1^1, v_1^2, \dots, v_1^m; v_2^1, v_2^2, \dots, v_2^m; \dots; v_n^1, v_n^2, \dots, v_n^m\}$ be the set of vertices as shown in Figure 13.

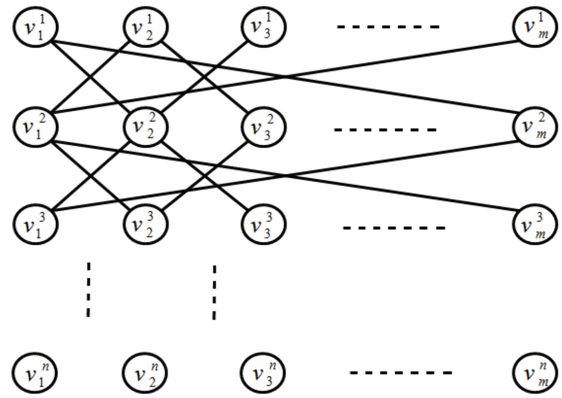


Fig. 13. The graph $P_n \wedge C_m$

If n is even, we define the labeling function $f : E(P_n \wedge C_m) \to \{0,1\}$ as follows:

$$f\left(v_{2j}^{2i} v_{2j-1}^{2i-1}\right) = f\left(v_{2j}^{2i} v_{2j-1}^{2i+1}\right) = f\left(v_{2j}^{2i} v_{2j+1}^{2i-1}\right) = f\left(v_{2j}^{2i} v_{2j+1}^{2i+1}\right) = 1;$$

$$1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, 1 \le j \le \frac{m}{2} - 1$$

$$f\left(v_m^{2i} v_1^{2i-1}\right) = f\left(v_m^{2i} v_1^{2i+1}\right) = f\left(v_m^{2i} v_{m-1}^{2i-1}\right) = f\left(v_m^{2i} v_{m-1}^{2i+1}\right) = 1;$$

$$1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1$$

If n is odd, then in addition to the labeling above we define

$$f\left(v_{2j}^{n} v_{2j-1}^{n-1}\right) = f\left(v_{2j}^{n} v_{2j+1}^{n-1}\right) = 1; 1 \le j \le \frac{m}{2} - 1$$

$$f\left(v_m^n v_1^{n-1}\right) = f\left(v_m^n v_{m-1}^{n-1}\right) = 1$$

And all the remaining edges will be labeled 0. The above labeling will induce vertex labeling as follows:

$$f\left(v_{2j}^{2i}\right) = f\left(v_{2j-1}^{2i-1}\right) = 1; \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, 1 \le j \le \frac{m}{2}$$

And all the remaining vertices will be labeled 0. From the above edge and vertex labelings, we will have $v_f(0) = v_f(1) = \frac{nm}{2}$, and $e_f(1) = e_f(0) = m(n-1)$. Therefore, $P_n \wedge C_m$ is an EPC graph.

Example 3.14: $P_5 \wedge C_6$ is an EPC graph as shown in Figure 14.

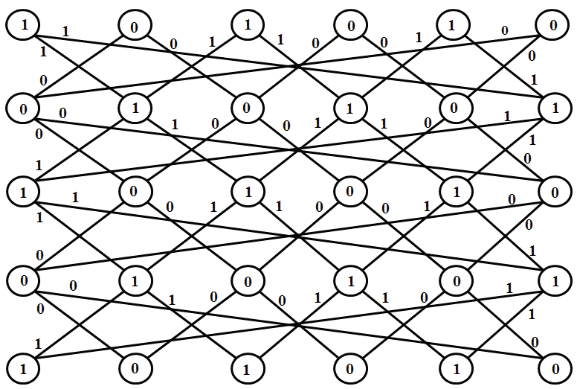


Fig. 14. An EPC labeling of the graph $P_5 \wedge C_6$

Theorem 3.15: The graph obtained from C_m by attaching a pendent path P_n to every vertex of C_m is an EPC graph.

Proof: This graph has $|V| = nm$ vertices and $|E| = nm$ edges. Let $\{v_1^1, v_1^2, \ldots, v_1^n; v_2^1, v_2^2, \ldots, v_2^n; \ldots; v_m^1, v_m^2, \ldots, v_m^n\}$ be the set of vertices.

For $k = \left\lfloor \frac{\left\lfloor \frac{nm}{2} \right\rfloor}{m} \right\rfloor$. If $\left\lfloor \frac{nm}{2} \right\rfloor = mk$, we define the labeling function $f : E \to \{0,1\}$ as follows:

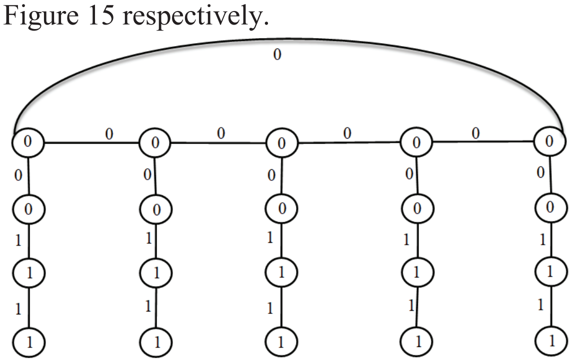
$$f\left(v_j^i v_j^{i+1}\right) = 1; \quad n-k \le i \le n-1, 1 \le j \le m$$

If $\left\lfloor \frac{nm}{2} \right\rfloor > mk$, then in addition to the labeling above we define

$$f\left(v_j^{n-k-1} v_j^{n-k}\right) = 1; \quad 1 \le j \le \left\lfloor \frac{nm}{2} \right\rfloor - mk$$

And all the remaining edges will be labeled 0. From the above edge labeling and according to the EPC labeling condition, we will have both $|v_f(0) - v_f(1)|$ *and* $|e_f(0) - e_f(1)|$ will be at most 1. Therefore, this graph is an EPC graph.

Example 3.16: The graph obtained from C_5 by attaching a pendent path P_4 and a pendent path P_5 to every vertex of C_5 are EPC graphs as shown in Figure 15 respectively.



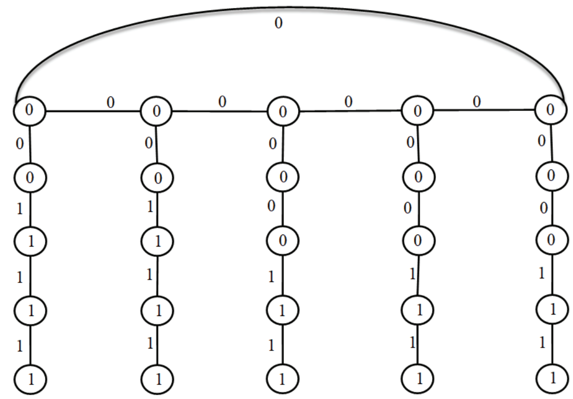


Fig. 15. An EPC labeling of the graph obtained from C_5 by attaching a pendent path P_4 and a pendent path P_5 to every vertex of C_5 respectively

4. ACKNOWLEDGEMENT

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