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Research Article

Raja General Function for Mixture Distributions and Tariq Distribution with Properties and Applications

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Abstract: Lifetime distributions play an important role in applied sciences. Keeping this consideration in this paper introduced a general function for mixture distribution, say as Raja General Function and discussed its some properties. Further using RGF introduced new Tariq distribution and conferred statistical properties including mean, variance, m.g.f, rth raw moments, reliability measures, order statistics, and graphical representation of functions also observed better goodness of fit than other lifetime data distribution functions using real-life data applications.

Keywords: Life Time Distribution, General Function, Reliability and RGF.

1. INTRODUCTION

There are several lifetime distributions have been developed for lifetime data in statistical literature like Weibull, Gamma, Exponential, lognormal, Lindley, Shanker, Sujatha, Amarendra, Akash and Devya distributions. These distributions are crucial in applied sciences like finance, insurance, biomedical and engineering sectors etc. Lifetime data distributions have relative characteristics to each other likewise Exponential, Wiebull, Lindley, Shanker, Aradhana, Akash, Amarendra and Sujatha distribution have expressed survival functions in open form and not required numerical integration but Gamma and lognormal. Lindley, Exponential, Lindley, Shanker, Aradhana, Akash, Amarendra, Sujatha and Devya are one parameter distribution, in these distributions the exponential has constant hazard rate others are monotonically increasing hazard rate and Amarendra is more flexible than others. The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of [1] Lindley (1958) distribution are respectively given as:

$$f_L(\mathrm{y},\theta) = \frac{\theta^2}{\theta+1}(1+y)e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.1}$$

$$F_L(\mathrm{y},\theta) = 1-\frac{\theta y+\theta+1}{\theta+1}e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.2}$$

The Lindley probability density function is a mixture of $f_1(\mathrm{y})$as an Exponential (θ) and $f_2(\mathrm{y})$ as a Gamma $(2,\theta)$ with mixture proportions $P_1=\frac{\theta}{\theta+1}$ and $P_2=\frac{1}{\theta+1}$ such as:

$$f_L(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) \tag{1.3}$$

[2]Ghitany *et al.* (2008) have discussed various properties of this distribution and showed that in many ways (1.1) provides a better model for some applications than the exponential distribution. The Lindley distribution has been modified, extended, mixed and generalized suiting their applications in different fields of knowledge by many researchers. [3] Zakerzadeh and Dolati (2009) introduced a generalized form of Lindley

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distribution, [4] Nadarajah *et al.* (2011) work on generalized Lindley distribution and their properties, [5] Ghitany *et al.* (2013) study on Power Lindley distribution and its associated inference, Computational Statistics and Data Analysis, [6] Abouammoh *et al.* (2015) introduced new generalized Lindley distribution and discussed its properties. [7] Shanker (2015 a) developed the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Akash distribution and provided it's different statistical and mathematical. [8] Shanker (2015 b) has provided Shanker distribution and its different statistical also mathematical properties, [9] Shanker (2016 d) has provided Poisson Shanker distribution which was derived from Poisson mixture of Shanker distribution and also provides its different statistical and mathematical properties, parameter estimation and its application for many count data sets. [10] Shanker (2016 e) introduced probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Aradhana distribution are given by

$$f_{Ard}(\mathrm{y},\theta) = \frac{\theta^3}{\theta^2+2\theta+2}(1+y)^2 e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.4)$$

$$F_{Ard}(\mathrm{y},\theta) = 1 - \left[1 + \frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right] e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.5)$$

The Aradhana probability density function is a mixture of $f_1(\mathrm{y})$as an Exponential (θ), $f_2(\mathrm{y})$ as a Gamma $(2,\theta)$ and $f_3(\mathrm{y})$ as a Gamma $(3,\theta)$ with mixture proportions $P_1 = \frac{\theta^2}{\theta^2+2\theta+2}$, $P_2 = \frac{2\theta}{\theta^2+2\theta+2}$ and $P_3 = \frac{2}{\theta^2+2\theta+2}$ such as:

$$f_{Ard}(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) + P_3 f_3(\mathrm{y}) \qquad (1.6)$$

[11] Shanker (2016 g) has introduced Sujatha distribution, [12] Shanker (2016 i) introduced probability density function and the cumulative distribution function of Amarendra distribution are given by

$$f_{Amd}(\mathrm{y},\theta) = \frac{\theta^4}{\theta^3+\theta^2+2\theta+6}(1+y+y^2+y^3)e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.7)$$

$$F_{Amd}(\mathrm{y},\theta) = 1 - \left[1 + \frac{\theta^3 y^3+\theta^2(\theta+3)y^2+\theta(\theta^2+2\theta+6)y}{\theta^3+\theta^2+2\theta+6}\right] e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.8)$$

The Amarendra probability density function is a mixture of $f_1(\mathrm{y})$as an Exponential (θ), $f_2(\mathrm{y})$ as a Gamma $(2,\theta)$, $f_3(\mathrm{y})$ as a Gamma $(3,\theta)$ and $f_4(\mathrm{y})$ as a Gamma $(4,\theta)$ with mixture proportions $P_1 = \frac{\theta^3}{\theta^3+\theta^2+2\theta+6}$, $P_2 = \frac{\theta^2}{\theta^3+\theta^2+2\theta+6}$, $P_3 = \frac{2\theta}{\theta^3+\theta^2+2\theta+6}$ and $P_4 = \frac{6}{\theta^3+\theta^2+2\theta+6}$ such as:

$$f_{Amd}(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) + P_3 f_3(\mathrm{y}) + P_4 f_4(\mathrm{y}) \qquad (1.9)$$

Further, p.d.f. and c.d.f of Devya distribution introduced by [13] Shanker (2016 k) given respectively as:

$$f_{Amd}(\mathrm{y},\theta) = \frac{\theta^5}{\theta^5+\theta^3+2\theta^2+6\theta+24}(1+y+y^2+y^3+y^4)e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.10)$$

$$F_{Amd}(\mathrm{y},\theta) = 1 - \left[1 + \frac{\theta^3 y^3+\theta^2(\theta+3)y^2+\theta(\theta^2+2\theta+6)y}{\theta^5+\theta^3+2\theta^2+6\theta+24}\right] e^{-\theta y} \quad y > 0;\ \theta > 0 \qquad (1.11)$$

The Devya probability density function is a mixture of $f_1(\mathrm{y})$as an Exponential (θ), $f_2(\mathrm{y})$ as a Gamma $(2,\theta)$, $f_3(\mathrm{y})$ as a Gamma $(3,\theta)$, $f_4(\mathrm{y})$ as a Gamma $(4,\theta)$ and $f_5(\mathrm{y})$ as a Gamma $(5,\theta)$ with mixture proportions $P_1 = \frac{\theta^5}{\theta^5+\theta^3+2\theta^2+6\theta+24}$, $P_2 = \frac{\theta^3}{\theta^5+\theta^3+2\theta^2+6\theta+24}$, $P_3 = \frac{2\theta^2}{\theta^5+\theta^3+2\theta^2+6\theta+24}$,$P_4 = \frac{6\theta}{\theta^5+\theta^3+2\theta^2+6\theta+24}$ and $P_5 = \frac{24}{\theta^5+\theta^3+2\theta^2+6\theta+24}$ such as:

$$f_{Amd}(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) + P_3 f_3(\mathrm{y}) + P_4 f_4(\mathrm{y}) + P_5 f_5(\mathrm{y}) \qquad (1.12)$$

For goodness of fit of Devya distribution over one parameter exponential, Lindley, Shanker, Akash, Aradhana, sujatha and Amarendra distribution have been used to examples of real lifetime data sets. [14] Shanker (2016) introduced Shambhu distribution with p.d.f and c.d.f defined as following;

$$f_{R5}(y) = \frac{\theta^6 e^{-\theta y}}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}[1 + y + y^2 + y^3 + y^4 + y^5] \quad y>0 \qquad (1.13)$$

$$F_{R5}(y) = 1 - \left[1 + \left\{\frac{\theta^5(y+y^2+y^3+y^4+y^5)+\theta^4(2y+3y+4y+5y)+\theta^3(6y+12y+20y^3)+\theta^2(24y+60y^2)+120\theta y}{\theta^5+\theta^4+2\theta^3+6\theta^2+24\theta+120}\right\}\right] e^{-\theta y}$$

$$\theta > 0; y > 0 \qquad (1.14)$$

2. RAJA GENERAL FUNCTION

The proposed General function of Probability density function (p.d.f.), namely Raja General Function (RGF) for mixture probability density function is defined as:

$$f_R(y, n, \theta) = \sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!} g(k + 1, \theta|y)\right] \quad \theta > 0; y > 0; n > 0 \qquad (2.1)$$

Here $f_R(y, n, \theta) = \sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!} g(k + 1, \theta|y)\right]$ $\theta > 0; y > 0; n > 0$ is gamma density function. If n=1 (2.1) is one parameter Lindley distribution, n=2 is Aradhana distribution, n=3 is Amarendra distribution, n=4 is Devya distribution, n=5 is Shambhu distribution. Further for n=6 here introduced as Tariq distribution and for further "n" could be introduced more distributions. The c.d.f of RGF one parameter θ for specified value of "n" is:

$$F_R(y, n, \theta) = \sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k + 1, \theta|y)dy\right] \quad \theta > 0; y > 0; n > 0 \qquad (2.2)$$

3. SOME PROPERTIES OF RGF

3.1 Moment generating function

Moment generating function of (2.1) is

$$M_x(t) = \frac{\theta^{n+1}\left[\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{k!}{\theta^{k+1}}\binom{m+k}{m}\left(\frac{t}{\theta}\right)^m\right]}{\sum_{k=0}^{n}\theta^{n-k}k!} \quad n > 0;\ \theta > 0 \qquad (3.1)$$

3.2 Moments

The rth Moment about origin of (2.1) is

$$\mu_r' = \frac{\sum_{k=0}^{n}(r+k)!\theta^{n-k}}{\theta^r\sum_{k=0}^{n}\theta^{n-k}k!} \qquad r = 1,2,3, \ldots \quad n>0 \qquad (3.2)$$

The first four moments about mean can be found with the following relations

$$\mu_2 = \mu_2' - (\mu_1')^2 \qquad (3.3)$$

$$\mu_3 = \mu_3' + 3\mu_2'\mu_1' - 2(\mu_1')^3 \qquad (3.4)$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4 \qquad (3.5)$$

3.3 Failure Rate Function

Let Y be a continuous random variable with p.d.f. (2.1) and c.d.f. (2.2). The hazard rate function known as the failure rate function is defined as:

$$h_R(y) = \lim_{\Delta y \to 0}\frac{P(Y<y+\Delta y|Y>y)}{\Delta y} = \frac{f(y)}{1-F(y)} \qquad (3.6)$$

$$h_R(y) = \frac{\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}g(k+1,\theta|y)\right]}{1-\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k+1,\theta|y)dy\right]} \qquad (3.7)$$

3.4 Survival function of RGF

The Survival function of random variable Y is given as:

$$S_R(y) = 1 - \sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k + 1, \theta|y)dy\right] \qquad (3.8)$$

3.5 Reversed hazard rate function of RGF:

$$H_R(y) = \frac{\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}g(k+1,\theta|y)\right]}{\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k+1,\theta|y)dy\right]} \qquad (3.9)$$

3.6 Cumulative hazard function of RGF:

$$CH_R(y) = -\ln\left|\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k+1,\theta|y)dy\right]\right| \quad (3.10)$$

3.7 Order Statistics of RGF

Let $y_{(1)}, y_{(2)}, \dots y_{(n)}$ be the order statistic of an independent and identically distributed (i.i.d), random sample $y_1, y_2, \dots y_n$. The c.d.f. and p.d.f. of ith order statistic are respectively given as following;

$$F_{y_{(i)}}(y;\ \theta, k) = \sum_{l=i}^{n}\sum_{j=0}^{n-l}\binom{n}{l}\binom{n-l}{j}(-1)^j\left[\sum_{k=0}^{n}\left\{\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y (k+ g\ (k+1,\theta|y)dy\right\}\right]^{j+l} \quad (3.11)$$

$$f(y_{(i)}) = \left[\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k+1,\theta|y)dy\right]\right]^{i-1}\left[1-\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}\int_0^y g(k+1,\theta|y)dy\right]\right]^{i-j}\sum_{k=0}^{n}\left[\frac{\theta^{n-k}.k!}{\sum_{k=0}^{n}\theta^{n-k}.k!}g(k+1,\theta|y)\right] \quad (3.12)$$

4. A NEW "TARIQ DISTRIBUTION"

For n=6 in RGF (2.1), we have Tariq distribution, which has the following p.d.f. and c.d.f respectively:

$$f_{R6}(y) = \frac{\theta^7 e^{-\theta y}}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}[1+y+y^2+y^3+y^4+y^5+y^6]\ \theta>0; \qquad y>0 \quad (4.1)$$

$$F_{R6}(y) = 1-[1+A]e^{-\theta y} \quad (4.2)$$

where

$$A = \left\{\frac{\begin{matrix}(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+\\(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6\end{matrix}}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\right\}$$

5. SOME PROPERTIES OF TARIQ DISTRIBUTION

5.1 Mean

The mean of random variable Y is defined as:

$$Mean = \frac{\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}{\theta(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} \quad (5.1)$$

5.2 Variance

The variance of random variable Y from Tariq distribution is obtained as follow:

$$Var(y) = \frac{\begin{pmatrix}\theta^{12}+4\theta^{11}+18\theta^{10}+96\theta^9+610\theta^8+4520\theta^7+35712\theta^6\\+30840\theta^5+504000\theta^4+192960\theta^3+\\1961280\theta^2+2056320\theta+3628800\end{pmatrix}}{\theta^2(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)^2} \quad (5.2)$$

5.3 Coefficient of Variance

The coefficient of variation and index of dispersion for random variable Y from Tariq distribution is obtained using equation (5.1) and (5.2) respectively as follow:

$$CV = \frac{\sqrt{\begin{pmatrix}\theta^{12}+4\theta^{11}+18\theta^{10}+96\theta^9+610\theta^8+4520\theta^7+35712\theta^6\\+30840\theta^5+504000\theta^4+192960\theta^3+\\1961280\theta^2+2056320\theta+3628800\end{pmatrix}}}{(\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040)} \quad (5.3)$$

$$\text{Dispersion Index} = \frac{\begin{pmatrix}\theta^{12}+4\theta^{11}+18\theta^{10}+96\theta^9+610\theta^8+4520\theta^7+35712\theta^6+\\30840\theta^5+504000\theta^4+192960\theta^3+\\1961280\theta^2+2056320\theta+3628800\end{pmatrix}}{\theta(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)*(\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040)} \quad (5.4)$$

5.4 Moments

Rth moment about origin put n=6 in (3.2),

$$\mu_r' = \frac{\sum_{k=0}^{6}(r+k)!\theta^{6-k}}{\theta^r\sum_{k=0}^{6}\theta^{6-k}k!} \qquad r = 1,2,3,\dots \quad (5.5)$$

1st four moment about origin are:

$$\mu_1' = \frac{\theta^6+2\theta^5+6\theta^4+24\theta^3+120\theta^2+720\theta+5040}{\theta(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} \quad (5.6)$$

$$\mu_2' = \frac{2(\theta^6+3\theta^5+12\theta^4+60\theta^3+360\theta^2+2520\theta+20160)}{\theta^2(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} \quad (5.5)$$

$$\mu_3' = \frac{6(\theta^6+4\theta^5+20\theta^4+120\theta^3+840\theta^2+6720\theta+60480)}{\theta^3(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} \quad (5.6)$$

$$\mu_4' = \frac{24(\theta^6+5\theta^5+30\theta^4+210\theta^3+1680\theta^2+15120\theta+151200)}{\theta^4(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} \quad (5.7)$$

5.5 Moment generating function

For Moment generating function for Tariq distribution put n=5 in (3.1),

$$M_y(t) = \frac{\theta^6\left[\sum_{m=0}^{\infty}\sum_{k=0}^{5}\frac{k!}{\theta^{k+1}}\binom{m+k}{m}\left(\frac{t}{\theta}\right)^m\right]}{\sum_{k=0}^{5}\theta^{5-k}k!} \quad (5.8)$$

5.6 Survival function

The Survival function for random variable Y from Tariq distribution is given following and graph in Figure 1 (d):

$$S_{R5}(y) = [1+B]e^{-\theta y} \quad (5.9)$$

$Whre$

$$B = \left\{\frac{(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6}{(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)}\right\}$$

5.7 Hazard rate function

The failure rate function also known as hazard function is given below and its graph is shown in Figure 1 (c):

$$h_{R5}(y) = \frac{f_{R5}(y)}{1-F_{R5}(y)} \quad (5.10)$$

$$h_{R5}(y) = \frac{\theta^6[1+y+y^2+y^3+y^4+y^5]}{\left\{\begin{matrix}(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)+\\(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+(\theta^2+5\theta+30)(y\theta)^4+\\(\theta+6)(y\theta)^5+(y\theta)^6\end{matrix}\right\}} \quad (5.11)$$

5.8 Reverse hazard function:

$$h_{R5}(y) = \frac{\theta^7e^{-\theta y}\{1+y+y^2+y^3+y^4+y^5\}}{\left[\begin{matrix}(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)-\\\{(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)+\\(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y\\+(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+(\theta^2+5\theta+30)(y\theta)^4+\\(\theta+6)(y\theta)^5+(y\theta)^6\}e^{-\theta y}\end{matrix}\right]} \quad (5.12)$$

5.9 Cumulate hazard rate function:

$$C_{R5}(y) = -\ln\left|1-\left[1+\left\{\frac{(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\right\}\right]e^{-\theta y}\right| \quad (5.13)$$

6. ORDER STATISTICS

The ordering of positive continuous random variables is an important tool for judging the comparative behavior. The i[th] order statistics with pdf of Tariq distribution given as follow;

$$f_{y_{(i)}}(y;\,\theta) = \frac{f(y)}{B(i,n-i+1)}[F(y)]^{i-1}[1-F(y)]^{n-i} \quad (6.1)$$

We know that $(1-x)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j$ and $B(i,n-i+1)$ is the beta function.

$$f_{y_{(i)}}(y;\,\theta) = \frac{f(y)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}[F(y)]^{i+j-1} \quad (6.2)$$

By putting eq. (4.1) and (4.2) in (6.2)

$$f_{(i)}(y;\,\theta) = \frac{\theta^7e^{-\theta y}[1+y+y^2+y^3+y^4+y^5+y^6]}{B(i,n-i+1)(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)} *$$

$$\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[1-\left[1+\left\{\frac{\begin{array}{c}(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+(\theta^2+5\theta+30)(y\theta)^4+\\(\theta+6)(y\theta)^5+(y\theta)^6\end{array}}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\right\}\right]e^{-\theta y}\right]^{i+j-1} \quad (6.3)$$

$$f_{(i)}(y;\ \theta)=\frac{\theta^7e^{-\theta y}[1+y+y^2+y^3+y^4+y^5+y^6]}{B(i,n-i+1)(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)}*$$

$$\sum_{j=0}^{n-i}\sum_{k=0}^{i+j-1}(-1)^{k+j}\binom{n-i}{j}\binom{i+j-1}{k}\left[1+\left\{\frac{\begin{array}{c}(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+\\(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6\end{array}}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\right\}\right]^k e^{-\theta ky} \quad (6.4)$$

The ith order statistics is:

$$F_{(i)}(y;\ \theta)=$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}(-1)^i\binom{n}{j}\binom{n-j}{i}\left\{1-\left[1+\left\{\frac{\begin{array}{c}(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+\\(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6\end{array}}{(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)}\right\}\right]e^{-\theta y}\right\}^{j+i} \quad (6.5)$$

$$F_{(i)}(y;\ \theta)=\sum_{j=0}^{n}\sum_{i=0}^{n-j}\sum_{k=0}^{i+j}(-1)^{k+i}\binom{n-i}{j}\binom{n}{j}\binom{i+j}{k}D \quad (6.6)$$

where

$$D=\left[1+\left\{\frac{\begin{array}{c}(\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)\theta y+\\(\theta^4+3\theta^3+12\theta^2+60\theta+360)(\theta y)^2+\\(\theta^3+4\theta^2+20\theta+120)(\theta y)^3+\\(\theta^2+5\theta+30)(y\theta)^4+(\theta+6)(y\theta)^5+(y\theta)^6\end{array}}{(\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720)}\right\}\right]^k e^{-\theta ky}$$

The c.d.f. of minimum and maximum order statistics of Tariq distribution can be obtained respectively by substituting i=1 and i=n in (6.6).

7. STOCHASTIC ORDERING

Stochastic ordering of positive continuous random variables is an important tool to observing their comparative or relative behavior. A random variable X is said to be smaller than a random variable Y if:

(1) Stochastic order $X \leq_{st} Y\ if\ F(X) \leq F(Y)\ \forall\ x$
(2) Hazard rate order $X \leq_{hr} Y\ if\ h(X) \leq h(Y)\ \forall\ x$
(3) Mean residual life order $X \leq_{mlr} Y\ if\ m(X) \leq m(Y)\ \forall\ x$

Table 1. The MLE's, Cramér-von Mises and Anderson-Darling statistics values.

Data set	**Distribution**	**Estimate**	**Anderson-Darling**	**Cramér-von Mises**	**Pearson χ^2**
1	Shambhu	0.08754	2.0947	0.34529	21.42
	Tariq	0.10218	1.33907	0.221483	18.56
2	Shambhu	0.19339	2.04269	0.309605	23.9677
	Tariq	0.22589	1.61868	0.233329	19.3226
3	Shambhu	1.83633	3.62614	0.52988	26.62
	Tariq	2.18007	2.59439	0.355952	22.72

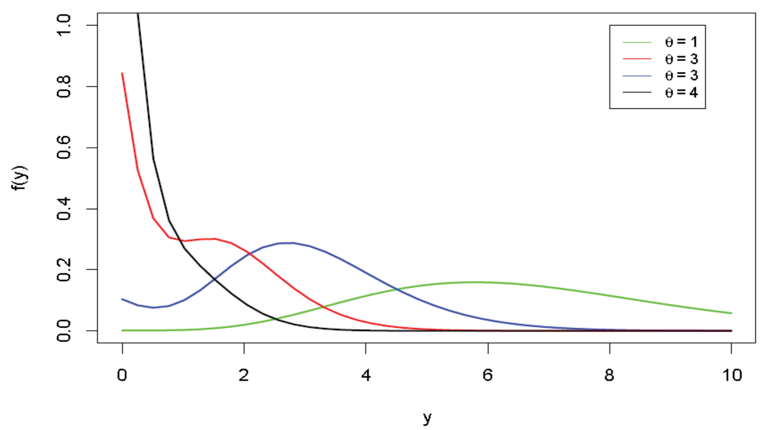

Fig. 1. (a) PDF of Tariq distribution different values of theta

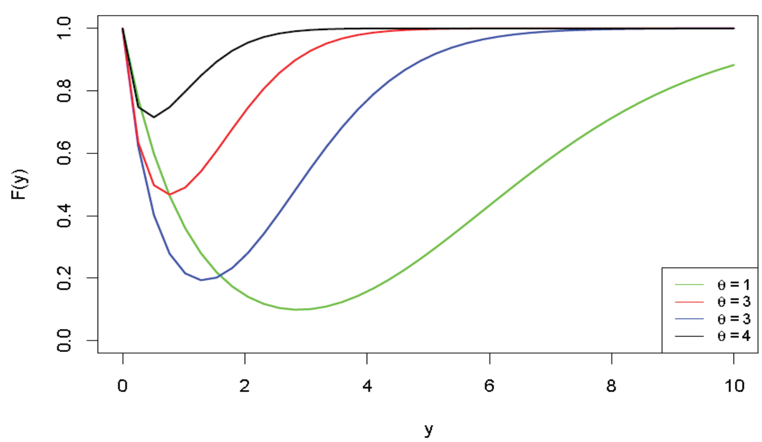

Fig. 1. (b) CDF of Tariq distribution for different values of theta

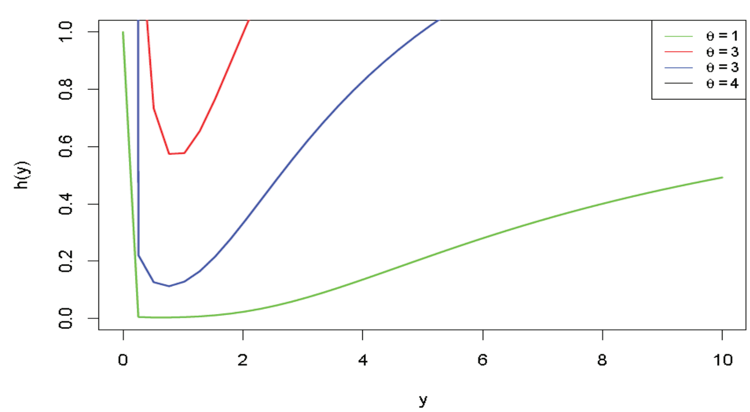

Fig. 1. (c) Hazard function of Tariq distribution flexibility for different values of theta

(4) Likelihood ratio order $X \leq_{lr} Y \; if \frac{f(X)}{f(Y)} \; decreasing \; in \; x$

Following well known relationship developing a stochastic ordering of distributions by [17] Shaked and Shanthikumar (1994).

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mlr} Y$$
$$\Downarrow$$
$$X \leq_{st} Y$$

The Tariq distribution (TD) is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem;

Theorem: Let $X \sim TD(\theta_1)$ and $Y \sim TD(\theta_2)$ if $\theta_1 > \theta_2$ then $\frac{f(X)}{f(Y)}$ as $X \leq_{lr} Y$, then their exist $X \leq_{lr} Y$, $X \leq_{hr} Y \; and \; X \leq_{mlr} Y$.

Proof: we have

$$\frac{f_X(y;\theta_1)}{f_Y(y;\theta_2)} = \frac{{\theta_1}^7 \begin{pmatrix} {\theta_1}^6 + {\theta_1}^5 + 2{\theta_1}^4 + 6{\theta_1}^3 + \\ 24{\theta_1}^2 + 120\theta_1 + 720 \end{pmatrix}}{{\theta_2}^7 \begin{pmatrix} {\theta_2}^6 + {\theta_2}^5 + 2{\theta_2}^4 + 6{\theta_2}^3 + \\ 24{\theta_2}^2 + 120\theta_2 + 720 \end{pmatrix}} e^{-(\theta_1-\theta_2)y}$$

$$\log\left\{\frac{f_X(y;\theta_1)}{f_Y(y;\theta_2)}\right\} = \log\left[\frac{{\theta_1}^7 \begin{pmatrix} {\theta_1}^6 + {\theta_1}^5 + 2{\theta_1}^4 + \\ 6{\theta_1}^3 + 24{\theta_1}^2 + 120\theta_1 + 720 \end{pmatrix}}{{\theta_2}^7 \begin{pmatrix} {\theta_2}^6 + {\theta_2}^5 + 2{\theta_2}^4 + \\ 6{\theta_2}^3 + 24{\theta_2}^2 + 120\theta_2 + 720 \end{pmatrix}}\right] - (\theta_1 - \theta_2)y$$

$$\frac{d}{dy}\log\left\{\frac{f_X(y;\theta_1)}{f_Y(y;\theta_2)}\right\} = -(\theta_1 - \theta_2)$$

Thus for $\theta_1 > \theta_2$, $\frac{d}{dy}\ln\left\{\frac{f_X(x;\theta_1,\alpha_1,k_1)}{f_Y(x;\theta_2,\alpha_2,k_2)}\right\} < 0$ it means that $X \leq_{lr} Y$ hence $X \leq_{lr} Y$, $X \leq_{hr} Y \; and \; X \leq_{mlr} Y$ which showing flexibility of Tariq distribution.

Maximum Likelihood Method

Let $y_1, y_2, \ldots y_n$ be a random sample from $f_{TD}(y;\, \theta, \alpha, k)$, then the Maximum Likelihood (ML) function is

$$L(y;\, \theta, \alpha, k) = \prod_{i=0}^{n} \frac{\theta^7 e^{-\theta y_i}}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}[1 + y_i + {y_i}^2 + {y_i}^3 + {y_i}^4 + {y_i}^5 + {y_i}^6] \quad (8.1)$$

Natural log of likelihood function is

$$lnL(y;\, \theta) = nln\left(\frac{\theta^7}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720}\right) + \sum_{i=0}^{n} \ln(1 + y_i + {y_i}^2 + {y_i}^3 + {y_i}^4 + {y_i}^5 + {y_i}^6) - \theta \sum_{i=0}^{n} y_i$$

$$\frac{dlnL(y;\theta)}{d\theta} = \frac{7n}{\theta} - \frac{n(6\theta^5+5\theta^4+8\theta^3+18\theta^2+48\theta+120)}{\theta^6+\theta^5+2\theta^4+6\theta^3+24\theta^2+120\theta+720} - \sum_{i=0}^{n} y_i \quad (8.2)$$

To find maximum likelihood estimate (MLE) put $\frac{dlnL(y;\theta)}{d\theta} = 0$ and determined following 7th degree polynomial equation;

$$\bar{x}\theta^7 + (\bar{x}-1)\theta^6 + 2(\bar{x}-1)\theta^5 + 6(\bar{x}-1)\theta^4 + 24(\bar{x}-1)\theta^3 + 120(\bar{x}-1)\theta^2 + 720(\bar{x}-1)\theta + -5040 = 0 \quad (8.3)$$

8. APPLICATIONS:

In order to compare lifetime distributions, K-S Statistics (Kolmogorov-Smirnov Statistics), Person Chi square and Anderson Darling statistics are applied on the data sets given below.

Kolmogorov-Smirnov statistics:

$$K - S = Sup_{(y)}|F_n(y) - F_0(y)|$$

$F_n(y)$= empirical distribution, $F_0(y) =$ hypothetical distribution; n= sample size

Person Chi square statistics:

$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$; O_i= Observed frequency, E_i=Expected frequency

Anderson Darling statistics:

$$A^2 = -n - S$$

Where $S = \sum_{i=1}^{n}(2i-1)/n[lnF(y_i) + \ln\{1 - F(y_{n+1-i})\}]$

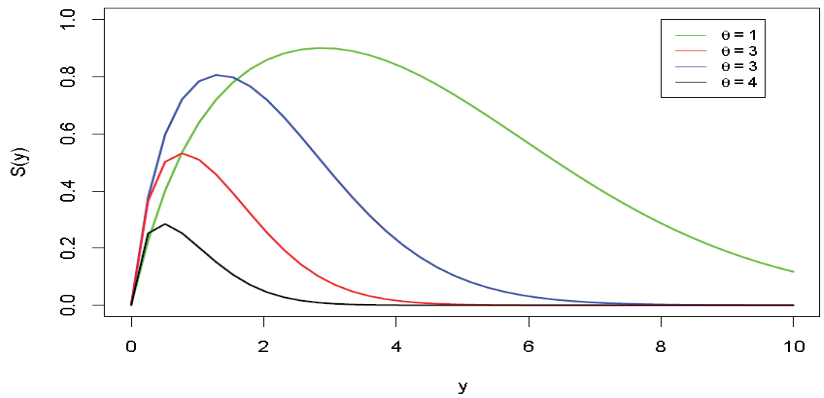

Fig. 1. (d) Survival function of Tariq distribution for different values of theta

values of theta

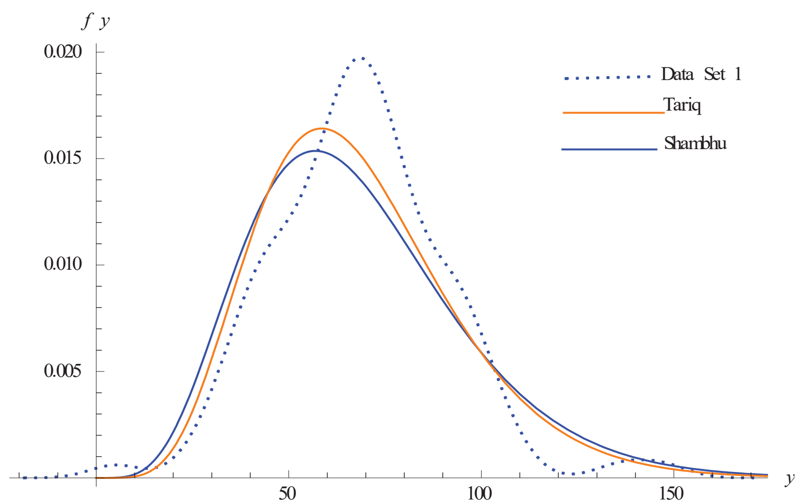

Fig. 2. Graph of distributions for data set 1.

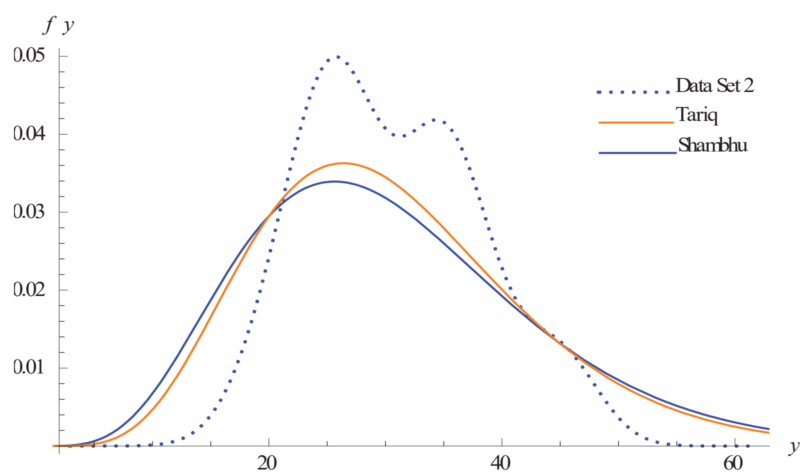

Fig. 3. Graph of distributions for data set 2.

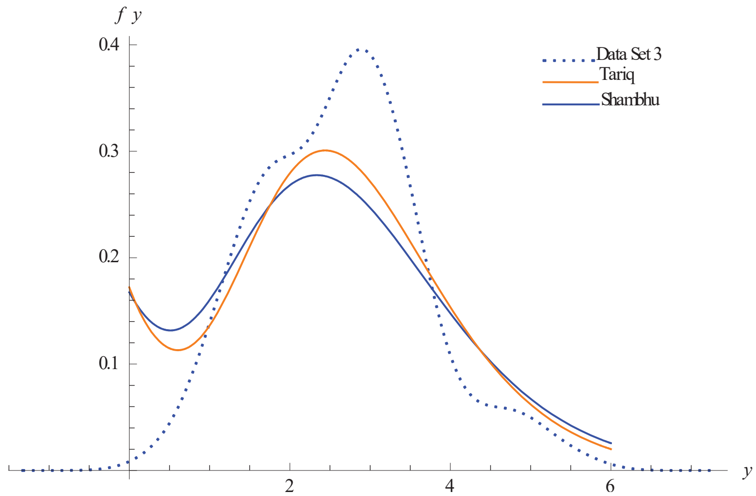

Fig. 4. Graph of distributions for data set 3.

F is the cumulative distribution function of the specified distribution and y_i are ordered data.

Data set 1 given by [16] Birnbaum and Saunders (1969) on the fatigue life of 6061 – T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data (×10^(-3)) are presented below (after subtracting 65).

Data set 1={5, 25, 31 ,32 ,34 ,35 ,38 ,39 ,39 ,40, 42, 43,43, 43, 44, 44, 47, 47, 48, 49, 49, 49 ,51, 54,55, 55, 55, 56, 56, 56, 58, 59, 59, 59, 59, 59,63, 63 ,64, 64, 65, 65, 65, 66, 66, 66, 66, 66,67, 67, 67, 68, 69, 69, 69, 69, 71, 71, 72, 73,73, 73, 74, 74 ,76, 76, 77, 77, 77, 77, 77, 77,79, 79, 80, 81, 83, 83, 84, 86, 86, 87, 90 ,91,92, 92, 92, 92, 93, 94, 97, 98, 98, 99, 101, 103,105, 109, 136 ,147};

The data set 2 is the strength data of glass of the aircraft window reported by [17] Fuller et al (1994) and is given as;

Data set 2 = {18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381};

The data set 3 is taken from [18] Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in Gba).

Data set 3 = {3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65};

9. CONCLUSION

In this study a general function for mixture distribution is proposed and named it as Raja General Function, which provide different probability distribution functions for different values of n. For n=1 it becomes Lindley distribution, for n=2 Ardhana distribution, for n=3 Amarendra distribution, for n=4 Devya distribution and for n=5 Shambhu distirbution Further for n=6, the distribution is proposed as Tariq distribution, and some of its properties like mean, variance, m.g.f, rth moment, survival function, hazard function, reversed hazard function, cumulative hazard function and order statistics are developed. Based on all the three real life data sets, the values of Anderson-Darling, Cramér-von Mises and Pearson χ^2 are computed for Tariq distribution as well as for Shambhu distribution and shown in Table 1, which is evident that the Tariq model better fitted the data than Shambhu distribution.

Note: The paper named "Raja General Function for Mixture distributions and Tariq Distribution" by taking from name of my younger brother Raja Asif Rasheed and my respected Colleague Mr. Tariq Jamshaid.

10. CONFLICT OF INTEREST

No conflict of interest.

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