Proceedings of the Pakistan Academy of Sciences:
A: Physical and Computational Sciences 58(4): 69-81 (2021)
Copyright © Pakistan Academy of Sciences
ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
https://doi.org/10.53560/PPASA(58-4)638

Pakistan Academy of Sciences

Research Article

A New Improved Classical Iterative Algorithm for Solving System of Linear Equations

Muhammad Shakeel Rind Baloch[1*], Zubair Ahmed Kalhoro[1], Mir Sarfraz Khalil[1], Prof. Abdul Wasim Shaikh[1]

[1]Institute of Mathematics and Computer Science, University of Sindh, Jamshoro, Pakistan

Abstract: The fundamental problem of linear algebra is to solve the system of linear equations (SOLE's). To solve SOLE's, is one of the most crucial topics in iterative methods. The SOLE's occurs throughout the natural sciences, social sciences, engineering, medicine and business. For the most part, iterative methods are used for solving sparse SOLE's. In this research, an improved iterative scheme namely, ''a new improved classical iterative algorithm (NICA)'' has been developed. The proposed iterative method is valid when the co-efficient matrix of SOLE's is strictly diagonally dominant (SDD), irreducibly diagonally dominant (IDD), M-matrix, Symmetric positive definite with some conditions and H-matrix. Such types of SOLE's does arise usually from ordinary differential equations (ODE's) and partial differential equations (PDE's). The proposed method reduces the number of iterations, decreases spectral radius and increases the rate of convergence. Some numerical examples are utilized to demonstrate the effectiveness of NICA over Jacobi (J), Gauss Siedel (GS), Successive Over Relaxation (SOR), Refinement of Jacobi (RJ), Second Refinement of Jacobi (SRJ), Generalized Jacobi (GJ) and Refinement of Generalized Jacobi (RGJ) methods.

Keywords: Diagonally Dominant, M-matrix, H-matrix, Irreducibly Diagonally Dominant, Refinement Jacobi, Generalized Jacobi, Rate of Convergence.

1. INTRODUCTION

The development of Jacobi method has been in focus for many researchers. For instance, the Refinement of Jabobi (RJ), Second Refinement of Jacobi (SRJ) and Generalized Jacobi (GJ) [1-5].

This research study has generalized the SRJ to a new improved classical iterative algorithm (NICA), where the bandness of diagonal of SRJ has been expended. It has been shown the NICA has minimized the number of iterations, increased the rate of convergence and reduced the spectral radius. The convergence of NICA has been proved for SDD, IDD, M-matrix, H-matrix and symmetric positive definite. In order to demonstrate the efficiency of NICA, different types of numerical examples are considered.

One of the most important topics in Numerical Linear Algebra is to solve a system of linear equations (SOLE's). With the increasing development of mathematical or scientific computing, the usage of SOLE's is also growing. There are many scientific computing problems which are converted into SOLE's. The basic problem of SOLE's is to solve $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a given invertible matrix, $b \in \mathbb{R}^n$ is a given vector and $x \in \mathbb{R}^n$ is the solution vector that is intended to find. The solution of SOLE's can be found by a direct or iterative method. A bunch of examples could be mentioned where system of linear equations arise are sparse. In such cases, the iterative methods are preferred. Among classical iterative methods, the method of successive over relaxation (SOR) has rapid convergence.

In recent years, many modifications in iterative methods like Gauss Jacobi, Gauss Seidel and Relaxation have been proposed in literature, which have either equal or better performance than these

Received: October 2021; Accepted: December 2021
*Corresponding Author: Muhammad Shakeel Rind Baloch <rindshakeel23@gmail.com>

methods. A simplest method to solve system of linear equations is Gauss Jacobi method.

From the various iterative methods available in the literature, the Jacobi's iterative method is mathematically simple. It is also independent of the number of unknowns i.e. the dimension of linear system. The clear-cut advantage is observed that, the round of errors is not collected from iteration to iteration but they are bounded to the last operation. Even though, Jacobi's method takes a lot of time to compute a very large dimensional linear system, its cost is the cheap and best choice for applying parallelization on its algorithm amongst iterative methods [6-7].

2. MATERIALS AND METHODS

The generalized Jacobi method, [1-2, 8, 11], is given by Splitting A into

$$A = T_m - E_m - F_m$$

The matrix $T_m = (a_{ij})$ is a banded matrix with band length $2m + 1$ and is defined as

$$T_m = \begin{cases} a_{ij}, |j - i| \le m \\ 0, otherwise \end{cases}$$

Where the matrices $-E_m$ and $-F_m$ are respectively strictly lower and upper triangular parts of

$A - T_m$, as shown below

$$T_m = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{bmatrix}$$

$$E_m = \begin{bmatrix} & & & \\ -a_{m+2,1} & & & \\ \vdots & \ddots & & \\ -a_{n,1} & \cdots & -a_{n,n-m-1} & \end{bmatrix}$$

$$F_m = \begin{bmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \end{bmatrix}$$

and

Now $Ax = b$ can be written as

$$(T_m - E_m - F_m)x = b \Rightarrow T_m x = (E_m + F_m)x + b$$

$$\Rightarrow x = T_m^{-1}(E_m + F_m)x + T_m^{-1}b$$

and if T_m^{-1} exists then

$$x^{k+1} = T_m^{-1}(E_m + F_m)x^{(k)} + T_m^{-1}b \text{ eq (1.1)}$$

The results of equation (1.1), is the matrix form of the generalized Jacobi iterative technique.

By introducing the notation

$T_{gj} = T_m^{-1}(E_m + F_m)$ and $c_{gj} = T_m^{-1}b$

The generalized Jacobi technique becomes in the form

$$x^{(k+1)} = T_{gj}x^{(k)} + c_{gj} \text{ eq(1.2)}$$

The Refinement of generalized Jacobi method, [1-2, 8], is given as follows:

$$T_m x = (E_m + F_m)x + b \Rightarrow T_m x = (T_m - A)x + b \Rightarrow x = x + T_m^{-1}(b - Ax)$$

after rearranging and simplifying, the equation becomes

$$x^{(k+1)} = \tilde{x}^{(k+1)} + T_m^{-1}\left(b - A\tilde{x}^{(k+1)}\right) \text{ eq(1.3)}$$

By using the equation (1.1) and substituting the value of $\tilde{x}^{(k+1)}$ into the equation (1.3) and simplifying, the equation is

$$x^{(k+1)} = [T_m^{-1}(E_m + F_m)]^2 + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b$$

Now by introducing the notation

$T_{rgj} = [T_m^{-1}(E_m + F_m)]^2$ and $c_{rgj} = [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b$

gives the refinement generalized Jacobi technique in the form

$x^{(k)} = T_{rgj}x^{(k-1)} + c_{rgj}$ for each k = 1, 2, 3,... .

2.1 Derivation of a New Improved Classical Iterative Algorithm

The new improved classical iterative algorithm (NICA) is proposed by using equation (1.4) and Substituting the value of $\tilde{x}^{(k+1)}$ into equation (1.3) and simplifying, i.e.

$$x^{(k+1)} = [T_m^{-1}(E_m + F_m)]^2 x^{(k)} + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b$$

$$x^{(k+1)} = \tilde{x}^{(k+1)} + T_m^{-1}\left(b - A\tilde{x}^{(k+1)}\right)$$

$$x^{(k+1)} = [T_m^{-1}(E_m + F_m)]^2 x^{(k)} + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b + T_m^{-1}\left(b - A\{[T_m^{-1}(E_m + F_m)]^2 x^{(k)} + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b\}\right)$$

the equation becomes

$$x^{(k+1)} = [T_m^{-1}(E_m + F_m)]^3 x^{(k)} + [I + T_m^{-1}(E_m + F_m) + \{T_m^{-1}(E_m + F_m)\}^2]T_m^{-1}b$$

Let $S_m = T_m^{-1}(E_m + F_m)$ and $R_m = T_m^{-1}b$

The new improved classical iterative algorithm (NICA) is proposed as

$$x^{(k+1)} = S_m^3 x^{(k)} + [I + S_m + S_m^2]R_m$$

Now by introducing the notation

$T_{nica} = S_m^3$ and $C_{nica} = [I + S_m + S_m^2]R_m$

Hence the standard form of new improved classical iterative algorithm is

$$x^{(k)} = T_{nica}x^{(k-1)} + C_{nica}$$

for each k = 1, 2, 3, ...

3. CONVERGENCE THEORY

Definition 1: ‘’A complex matrix $A \in \mathbb{C}^{n\times n}$ is reducible if and only if there exists a permutation matrix p (i.e. p is obtained from the identity matrix I by a permutation of the rows of identity matrix) and an integer $k \in \{1,2,\dots,n-1\}$ such that

$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ Where A_{11} is $k \times k$ and A_{22} is $(n-k)\times(n-k)$. If A is not reducible, then it is said to be irreducible’’ [12].

Definition 2: ’’An $n \times n$ matrix $A = (a_{ij})$ is said to be diagonally dominant (DD) if

$|a_{i,i}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,i}|, (1 \leq i \leq n)$‘’ [11,13].

Definition 3: ’’An $n \times n$ matrix $A = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if $|a_{i,i}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,i}|, (1 \leq i \leq n)$’’ [11, 13].

Definition 4: ’’A irreducibly diagonally dominant (IDD) if A is irreducible and diagonally dominant, with strict inequality holding in Definition 3 for at least one i'’ [11-12].

Definition 5: ‘’The spectral radius of matrix A is the largest absolute value of the eigenvalues of A**:** $\rho(A) = max\{|\lambda|: \lambda \in \sigma(A)\}$’’ [12].

Lemma 1: ‘’The spectral radius satisfies the following rules

$\rho(kA) = |k|\rho(A)$ for all $k \in \mathbb{C}$ and $A \in \mathbb{C}^{n\times n}$.

$\rho\left(A^k\right) = (\rho(A))^k$ for all $\mathbb{N}$ and $A \in \mathbb{C}^{n\times n}$ ‘’ [3].

Definition 6: An $n \times n$ matrix $A = (a_{ij})$ is said to be symmetric positive definite (SPD) if A is symmetric, ($A = A^T$) and positive definite $x^T A x > 0$ for all $a \neq 0$ [11].

Note: In SPD every $a_{ii} > 0$ for $i = 1, 2, \dots, n$.

Definition 7: A matrix $Z = (z_{ij})$ is said to be Z-matrix if $z_{ij} \leq 0$ for $i \neq j$ [14].

Definition 8: ‘’A matrix is said to be an M-matrix if it satisfies the following four properties:

1) $a_{ii} > 0$ for $i = 1, 2, \dots, n$

2) $a_{ii} \leq 0$ for $i \neq j,\ i, j = 1, 2, \dots, n$

3) A is nonsingular

4)$A^{-1} \geq 0$.

Alternatively, A matrix $A \in \mathbb{R}^{n\times n}$ is said to be nonsingular M-matrix if A can be written as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$‘’ [12].

Definition 9: A matrix $A \epsilon \mathbb{C}^{n\times n}$ is said to be an H-matrix if its comparison matrix $H(A) = (m_{ij})$ with $m_{ii} = |a_{ii}|$ and $m_{ij} = -|a_{ij}|$, is an M-matrix [12].

Definition 10: ‘’Let $A = (a_{i\,j}) \in \mathbb{R}^{n\times n}$, $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called:

a) Convergent if $\rho(M^{-1}N) < 1$.

b) Regular if $M^{-1} \geq 0$ and $N \geq 0$

c) Nonnegative if $M^{-1}N \geq 0$

(d) M-splitting if M is a nonsingular M-matrix and $N \geq 0$‘’ [12].

Definition 11: ‘’The rate of convergence of an iterative method is $R(T) = -\log 10\,[\rho(T)]$ or $\tau = -\ln \rho$. This rate of convergence is also called asymptotic rate of convergence’’ [11, 15].

Theorem 1: ‘’A linear iteration $\Phi(x, b) = Mx + Nb$ with the iteration matrix $M = M[A]$ is convergent if and only if $\rho(M) < 1$’’ [16].

Theorem 2: ‘’Let $A = M - N$ be a regular splitting of the matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$’’ [16].

Theorem 3: ‘’Let $A = (a_{i\,j})$, $B = (b_{i\,j})$ be two matrices such that $A \leq B$ and $b_{ij} \leq 0$ for all $i \neq j$. Then if A is an M-matrix, so is the matrix B’’ [16].

Theorem 4: ‘’The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n x = 0$, for every x’’[13].

Theorem 5: ‘’Let $A = (a_{i\,j})$ be a $n \times n$ strictly or irreducibly diagonally dominant complex matrix. Then, the matrix A is non-singular. If all the diagonal entries of A are in positive real numbers, then eigenvalues of λ_i of A satisfy $Re(\lambda_i) > 0$, $1 \leq i \leq n$’’ [12].

Theorem 6: ‘’Let A and B be two $n \times n$ matrices with $O \leq |B| \leq A$ then

$\rho(B) \leq \rho(A)$’’ [12].

Theorem 7: ‘’Let M, N be a regular splitting of a matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative’’ [16].

Lemma 2: ‘’Let $M = (m_{ij})$ and $N = (n_{ij})$ be $n \times n$ matrices with M being strictly diagonally dominant. Then the eigenvalues of $M^{-1}N$ satisfy ‘’ [12].

Theorem 8: ‘’Let A be strictly diagonally dominant matrix. Then for any natural number $m \leq n$, the Generalized Gauss Jacobi method is convergent for any initial guess $x^{(0)}$’’ [8].

Theorem 9: If A and $2T_m - A$ are symmetric and positive definite matrices, then Generalized Jacobi is convergent for any initial guess.

Proof*:* Let A and $2T_m - A$ be SPD. We know $x^*Ax > 0$ and $x^*(2T_m - A)x > 0$, where $A = T_m - E_m - {E_m}^T$.

Taking the associative generalized Jacobi we have

$${T_m}^{-1}(E_m + {E_m}^T)x = \lambda x \Rightarrow (E_m + {E_m}^T)x = \lambda T_m x$$

$\Rightarrow x^*(E_m + {E_m}^T)x = \lambda x^* T_m x$

$\Rightarrow x^*(T_m - A)x = \lambda x^* T_m x \Rightarrow x^* T_m x - x^* A x = \lambda x^* T_m x \Rightarrow x^* T_m x - \lambda x^* T_m x = x^* A x$

$\Rightarrow x^* T_m x(1-\lambda) = x^* A x$ equation (1)

as $x^* A x > 0$ so $x^* T_m x > 0$ therefore $(1-\lambda) > 0$

$\Rightarrow 1 > \lambda$ *or* $\lambda < 1$ Equation (2)

Now $x^*(2T_m - A)x > 0 \Rightarrow 2x^* T_m x - x^* A x > 0 \Rightarrow 2x^* T_m x > x^* A x$ by using equation (1) we get

$$2x^* T_m x > x^* T_m x(1-\lambda) \Rightarrow 2x^* T_m x - x^* T_m x(1-\lambda) > 0$$

$$\Rightarrow 2x^* T_m x - x^* T_m x + \lambda x^* T_m x > 0 \Rightarrow x^* T_m x + \lambda x^* T_m x > 0 \Rightarrow x^* T_m x(1+\lambda) > 0$$

As $x^* T_m x > 0$ *so* $(1+\lambda) > 0 \quad \Rightarrow \lambda > -1$ Equation (3)

By taking (2) and (3) we have $-1 < \lambda < 1$, where λ an Eigenvalue of ${T_m}^{-1}(E_m + {E_m}^T)$ Hence $\rho\left({T_m}^{-1}(E_m + {E_m}^T)\right) < 1$.

Theorem 10: Let A be a Z-matrix, then the following statements are equivalent: [17].

a) A is non-singular M-matrix.
b) There is a positive vector x such that $Ax \gg 0$
c) All principal sub-matrices of A are non-singular M-matrices.
d) All principal minors are positive

Theorem 11: Let A be a matrix with off-diagonal entries are non-positive and A is nonsingular, then the following statements are equivalent: [18]

i) A is an M-matrix.
ii) A is semi positive, that is, there exists $x > 0$ such that $Ax > 0$.
iii) A is monotone, that is, $Ax \geq 0$ implies $x \geq 0$.
iv) $A^{-1} \geq 0$.

Theorem 12: If $A \neq 0$, $x \geq 0$, and $x \neq 0$. If $Ax \geq \alpha x$, for some real α, then $\rho(A) \geq \alpha$ [19].

Theorem 13: If A is an H-matrix, then GJ converges for any initial guess $x^{(0)}$.

Proof: Let A be an H-matrix. Consider of A as $A = T_m - (E_m + F_m)$ for some m. Let $M = T_m = D + R_m$, and $N = E_m + F_m$, where D=diag(A), and $R_m = T_m - D$. Let $H(A)$ be the comparison matrix of A, so that $H(A)$ is an M-matrix.

Note that $H(A) = |D| - |R_m| - |E_m| - |F_m|$. Then $H(A) = M_1 - N_1$ is the generalized Jacobi splitting of $H(A)$, where $M_1 = |D| - |R_m|$ and $N_1 = |E_m| + |F_m|$. Hence by theorem 8 , $\rho(M_1^{-1}N_1) < 1$. Let λ be any eigenvalue of $M^{-1}N$, and let $x \neq 0$ such that $M^{-1}Nx = \lambda x$, that is $Nx = \lambda Mx$, then $|Nx| = |\lambda Mx| \quad \Rightarrow |\lambda||Mx| = |Nx| \leq |N||x|$ implies that

$|\lambda||Dx + R_m x| \leq |N||x|$ Equation (1). Again

$|Dx + R_m x| = |Dx - (-R_m x)| \geq \big||Dx| - |R_m x|\big| \geq |Dx| - |R_m x| = |D||x| - |R_m x|$. Now equation (1) implies that $|\lambda|(|D||x| - |R_m x|) \leq |N||x| \Rightarrow |\lambda|(|D||x| - |R_m||x|) \leq |N||x|$
$\therefore |R_m x| \leq |R_m||x|$. Now $|N| = |E_m + F_m| \leq |E_m| + |F_m| = N_1 \Rightarrow |\lambda|M_1|x| \leq N_1|x|$ equation (2). Since M_1 is an Z-matrix and $H(A) \leq M_1$ so by theorem 10, M_1 is an invertible M-matrix and hence by theorem 11, $M_1^{-1} \geq 0$. equation (2) implies that $|\lambda||x| \leq M_1^{-1}N_1|x|$. as $M_1^{-1}N_1 \geq 0$, $|x| \geq 0$, and $x \neq 0$, so by theorem 12, $|\lambda| \leq \rho(M_1^{-1}N_1) < 1$. This shows that $\rho(M^{-1}N) < 1$, and GJ method converges.

Theorem 14: ‘’If A is strictly diagonally dominant matrix, then the refinement of generalized Jacobi method converges for any choice of the initial approximation $x^{(0)}$’’ [2].

Theorem 15: If A and $2T_m - A$ are symmetric and positive definite matrices, then Refinement of Generalized Jacobi is convergent for any initial guess.

Proof: Using Equation (1.1) and Theorem 1, we have $\rho(T_m^{-1}(E_m + F_m)) < 1$.

Since A is SPD matrix. Let X be the exact solution of $Ax = b$.Then generalized Jacobi can be written as $X = T_m^{-1}(E_m + {E_m}^T)X + T_m^{-1}b \Rightarrow X - T_m^{-1}(E_m + E_m^T)X = T_m^{-1}b$

$\Rightarrow X[I - T_m^{-1}(E_m + E_m^T)] = T_m^{-1}b \Rightarrow X = [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b.$

Using equation (1.4) and exact solution X we have:

$$X = [T_m^{-1}(E_m + E_m^T)]^2X + [I + T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$$

$$X = [I - \{T_m^{-1}(E_m + E_m^T)\}^2]^{-1}[I + T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$$

After expanding and multiplying we get

$$X = [I + \{T_m^{-1}(E_m + E_m^T)\}^2 + \{T_m^{-1}(E_m + E_m^T)\}^4 + \cdots][I + T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$$

$$X = [I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2 + \{T_m^{-1}(E_m + E_m^T)\}^3 + \{T_m^{-1}(E_m + E_m^T)\}^4 \ldots]T_m^{-1}b$$

$$= [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b = X$$

Therefore $X = [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$ is consistent to $Ax = b$ and generalized Jacobi. On the other hand

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^2x^{(k)} + [I + T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$$

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^4x^{(k-1)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^3]^{-1}T_m^{-1}b$$

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^6x^{(k-2)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^5]^{-1}T_m^{-1}b$$

$$= \vdots\; x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^{2(k+1)}x^{(0)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^{2n+1}]^{-1}T_m^{-1}b$$

We have $\rho\big(T_m^{-1}(E_m + E_m^T)\big) < 1$ since A is SPD

$$\Rightarrow \lim_{k\to\infty}[T_m^{-1}(E_m + E_m^T)]^{2(k+1)}x^{(0)} = 0$$

$$\Rightarrow \lim_{k\to\infty}x^{(k+1)} = \lim_{k\to\infty}[T_m^{-1}(E_m + E_m^T)]^{2(k+1)}x^{(0)} + \textstyle\sum_{k=0}^{\infty}[T_m^{-1}(E_m + E_m^T))]^kT_m^{-1}b$$

$$= 0 + [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b \longrightarrow X$$

$\Rightarrow\ \rho(\{T_m^{-1}(E_m + E_m^T)\}^2) < 1 = \rho\big(T_m^{-1}(E_m + E_m^T)\big)^2 < 1$. Therefore, refinement of generalized Jacobi (RGJ) is convergent.

Theorem 16: If A is an H-matrix, then RGJ converges for any initial guess $x^{(0)}$.

Proof: by using theorem 8, $\rho(M_1^{-1}N_1) < 1$. For RGJ we have $\rho((M_1^{-1}N_1)^2) < 1$, it implies by lemma 1 ,$\rho(M_1^{-1}N_1)^2 < 1$. It follows from theorem 13 that $|\lambda| \le \rho(M_1^{-1}N_1)^2 < 1$. This shows that $\rho(M^{-1}N)^2 < 1$. Hence RGJ method converges.

Theorem 17: If A is strictly diagonally dominant matrix, then new improved classical iterative algorithm (NICA) converges for any choice of the initial approximation $x^{(0)}$.

Proof: Let x be the real exact solution of linear system $Ax = b$. We know that if A is strictly diagonally dominant matrix and Refinement Generalized Jacobi method is $\tilde{x}^{(k+1)} = [T_m^{-1}(E_m + F_m)]^2x^{(k)} + [I + T_m^{-1}(E_m + F_m)]T_m^{-1}b$

Then using theorems 8 and 14, the Generalized Jacobi and Refinement Generalized Jacobi methods are convergent.

So $\tilde{x}^{k+1} \to x$

We can write the new improved classical iterative algorithm (NICA) as

$$x^{(k+1)} = \tilde{x}^{(k+1)} + T_m^{-1}(b - A\tilde{x}^{(k+1)})$$

$$\|x^{(k+1)} - x\| = \|\tilde{x}^{(k+1)} - x + T_m^{-1}\left(b - A\tilde{x}^{(k+1)}\right)\|$$

$$\|x^{(k+1)} - x\| \le \|\tilde{x}^{(k+1)} - x\| + \|T_m^{-1}\left(b - A\tilde{x}^{(k+1)}\right)\|$$

$$\|x^{(k+1)} - x\| \le \left\|\tilde{x}^{(k+1)} - x\right\| + \|T_m^{-1}\|\|\left(b - A\tilde{x}^{(k+1)}\right)\|$$

It implies that

$$= 0 + \|T_m^{-1}\|\|b - b\| = 0 + 0 \Rightarrow \|x - x\|\|T_m^{-1}\|\|b - Ax\|$$

Then by using theorem 4, $\rho[T_m^{-1}(E_m + F_m)]^3 = (\rho[T_m^{-1}(E_m + F_m)])^3$

Hence, completes the proof.

Theorem 18: The new improved classical iterative algorithm (NICA) converges faster than the generalized Jacobi and refinement of generalized Jacobi method when generalized Jacobi method is convergent.

Proof: Let

$G_m = T_m^{-1}(E_m + F_m),\ \ C_m = T_m^{-1}b,$

$B_m = [T_m^{-1}(E_m + F_m)]\, T_m^{-1}b$ and $K_m = [I + T_m^{-1}(E_m + F_m) + \{T_m^{-1}(E_m + F_m)\}^2]T_m^{-1}b.$

Then we can write the equations of generalized Jacobi, Refinement generalized Jacobi and a new improved classical iterative algorithm (NICA) as

$x^{(k+1)} = G_m x^{(k)} + C_m,$

$x^{(k+1)} = G_m^2 x^{(k)} + B_m$ and

$x^{(k+1)} = G_m^3 x^{(k)} + K_m$ respectively.

Given that $||G_m|| < 1$.

Let x be the exact solution of $Ax = b$.

Let us consider generalized Jacobi method:

$$x^{(k+1)} = G_m x^{(k)} + C_m \Rightarrow x^{(k+1)} - x = G_m x^{(k)} - x + C_m$$

Adding and subtracting $G_m x$ on R.H.S

$$x^{(k+1)} - x = G_m\left(x^{(k)} - x\right) + G_m x + C_m - x \Rightarrow x^{(k+1)} - x = G_m\left(x^{(k)} - x\right) \therefore (G_m x + C_m = x)$$

$$\|x^{(k+1)} - x\| \le \|G_m\left(x^{(k)} - x\right)\| \le \|G_m\|\|\left(x^{(k)} - x\right)\|$$

$$\|G_m\|\left\|\left(x^{(k)} - x\right)\right\| \le \|G_m^2\|\left\|\left(x^{(k-1)} - x\right)\right\| \le \|G_m^3\|\left\|\left(x^{(k-2)} - x\right)\right\| \le \cdots \le \|G_m^n\|\left\|\left(x^{(1)} - x\right)\right\|$$

It implies that

$$\|x^{(k+1)} - x\| \le \|G_m^n\|\left\|\left(x^{(1)} - x\right)\right\| \le \|G_m\|^n\left\|x^{(1)} - x\right\| \ (\therefore \|A^n\| \le \|A\|^n)$$

$$\left\|x^{(k+1)} - x\right\| \le \|G_m\|\left\|x^{(1)} - x\right\| \ eq(a)$$

Now let us consider Refinement of Generalized Jacobi method:

$$x^{(k+1)} = G_m^2 x^{(k)} + B_m \Rightarrow x^{(k+1)} - x = G_m^2 x^{(k)} + B_m - x$$

Adding and subtracting $G_m^2 x$

$$x^{(k+1)} - x = G_m^2(x^{(k)} - x) + G_m^2 x + B_m - x \Rightarrow x^{(k+1)} - x = G_m^2(x^{(k)} - x) \ (\therefore G_m^2 x + B_m = x)$$

$$\|x^{(k+1)} - x\| \le \|G_m^2\left(x^{(k)} - x\right)\| \le \|G_m^2\|\|\left(x^{(k)} - x\right)\|$$

$$\|G_m^2\|\left\|\left(x^{(k)} - x\right)\right\| \le \|G_m^4\|\left\|\left(x^{(k-1)} - x\right)\right\| \le \|G_m^6\|\left\|\left(x^{(k-2)} - x\right)\right\| \le \cdots \le \|G_m^{2n}\|\left\|\left(x^{(1)} - x\right)\right\| \le \|G_m\|^{2n}\left\|\left(x^{(1)} - x\right)\right\|$$

$$\|x^{(k+1)} - x\| \le \|G_m\|^{2n}\left\|\left(x^{(1)} - x\right)\right\| \ eq(b)$$

Again let us consider the new improved classical iterative algorithm (NICA):

$$x^{(k+1)} = G_m^3 x^{(k)} + K_m \Rightarrow x^{(k+1)} - x = G_m^3 x^{(k)} - x + K_m$$

Adding and Subtracting $G_m^3 x$ on R.H.S

$$x^{(k+1)} - x = G_m^3(x^{(k)} - x) + G_m^3 x + K_m - x \Rightarrow x^{(k+1)} - x = G_m^3(x^{(k)} - x) \ (\because G_m^3 x + K_m = x)$$

$$\|x^{(k+1)} - x\| = \|G_m^3(x^{(k+1)} - x)\| \leq \|G_m^3\|\|(x^{(k+1)} - x)\| \leq \|G_m^6\|\|(x^{(k)} - x)\| \leq \cdots \leq \|G_m^{3n}\|\|(x^{(1)} - x)\| \leq \|G\|^{3n}\|(x^{(1)} - x)\|$$

$$\|x^{(k+1)} - x\| \leq \|G\|^{3n}\|(x^{(1)} - x)\| \quad eq(c)$$

According to the coefficients of the above inequalities eq(a), eq(b) and eq(c), we have

$\|G_m\|^{3n} \leq \|G_m\|^{2n} \leq \|G_m\|^n$ since $\|G\| < 1$

Hence, the new improved classical iterative algorithm (NICA) is converges faster than Generalized Jacobi and Refinement generalized Jacobi.

Theorem 19: If generalized Jacobi method is convergent, then a new improved classical iterative algorithm (NICA) is convergent.

Proof: Given:» Generalized Jacobi method is convergent.

Required:» The new improved classical iterative algorithm (NICA) is convergent.

We know that generalized Jacobi method is convergent

$\Leftrightarrow \rho[T_m^{-1}(E_m + F_m)] < 1.$

We want to show that the new improved classical iterative algorithm (NICA) is convergent, using lemma (1) $[\rho\{T_m^{-1}(E_m + F_m)\}]^3 \leq \rho\{T_m^{-1}(E_m + F_m)\}^3 < 1.$

Hence the new improved classical iterative algorithm (NICA) is convergent when the generalized Jacobi method is convergent.

Theorem 20: If A and $2T_m - A$ are symmetric and positive definite matrices, then NICA is convergent for any initial guess.

Proof: Using Equation (1.1) and Theorem 1, we have $\rho(T_m^{-1}(E_m + F_m)) < 1$.

Since A is SPD matrix. Let X be the exact solution of $Ax = b$. Then refinement of generalized Jacobi can be written as

$$X = T_m^{-1}(E_m + E_m^T)X + T_m^{-1}b \Rightarrow X - T_m^{-1}(E_m + E_m^T)X = T_m^{-1}b$$

$\Rightarrow X[I - T_m^{-1}(E_m + E_m^T)] = T_m^{-1}b \Rightarrow X = [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$ if $x^{(k+1)} \longrightarrow X$.

Using equation (1.6) and exact solution X we have:

$$X = [T_m^{-1}(E_m + E_m^T)]^3 X + [I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2]^{-1}T_m^{-1}b$$

$$X = [I - \{T_m^{-1}(E_m + E_m^T)\}^3]^{-1}[I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2]^{-1}T_m^{-1}b$$

After expanding and multiplying we get

$$X = [I + \{T_m^{-1}(E_m + E_m^T)\}^3 + \{T_m^{-1}(E_m + E_m^T)\}^6 + \cdots][I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2]^{-1}T_m^{-1}b$$

$$X = [I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2 + \{T_m^{-1}(E_m + E_m^T)\}^3 + \{T_m^{-1}(E_m + E_m^T)\}^4 \ldots]T_m^{-1}b$$

$$= [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$$

Therefore $X = [I - T_m^{-1}(E_m + E_m^T)]^{-1}T_m^{-1}b$ is consistent to $Ax = b$ and refinement of generalized Jacobi. On the other hand

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^3 x^{(k)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \{T_m^{-1}(E_m + E_m^T)\}^2]^{-1}T_m^{-1}b$$

$\Rightarrow x^*(\mathrm{E_m} + \mathrm{E_m}^{\mathrm{T}})x = \lambda x^* \mathrm{T_m} x$

$\Rightarrow x^* (\mathrm{T_m} - A) x = \lambda x^* \mathrm{T_m} x \Rightarrow x^* \mathrm{T_m} x - x^* \mathrm{A} x = \lambda x^* \mathrm{T_m} x \Rightarrow x^* \mathrm{T_m} x - \lambda x^* \mathrm{T_m} x = x^* \mathrm{A} x$

$\Rightarrow x^* \mathrm{T_m} x (1 - \lambda) = x^* \mathrm{A} x$ equation (1)

as $x^* \mathrm{A} x > 0$ so $x^* \mathrm{T_m} x > 0$ therefore $(1 - \lambda) > 0$

$\Rightarrow 1 > \lambda$ *or* $\lambda < 1$ Equation (2)

Now $\mathrm{x^*(2T_m - A)x} > 0 \Rightarrow \mathrm{2x^*T_m x - x^*Ax} > 0 \Rightarrow \mathrm{2x^*T_m x} > \mathrm{x^*Ax}$ by using equation (1*)* we get

$$\mathrm{2x^*T_m x} > x^* \mathrm{T_m} x (1 - \lambda) \Rightarrow \mathrm{2x^*T_m x} - x^* \mathrm{T_m} x (1 - \lambda) > 0$$

$$\Rightarrow \mathrm{2x^*T_m x} - x^* \mathrm{T_m} x + \lambda x^* \mathrm{T_m} x > 0 \Rightarrow \mathrm{x^*T_m x} + \lambda x^* \mathrm{T_m} x > 0 \Rightarrow \mathrm{x^*T_m x}(1 + \lambda) > 0$$

As $\mathrm{x^*T_m x} > 0$ *so* $(1 + \lambda) > 0 \quad \Rightarrow \lambda > -1$ Equation (3*)*

By taking (2) and (3*)* we have $-1 < \lambda < 1$, where λ an Eigenvalue of $\mathrm{T_m}^{-1}(\mathrm{E_m} + \mathrm{E_m}^{\mathrm{T}})$ Hence $\rho\left(\mathrm{T_m}^{-1}(\mathrm{E_m} + \mathrm{E_m}^{\mathrm{T}})\right) < 1$.

Theorem 10: Let A be a Z-matrix, then the following statements are equivalent: [17].

a) A is non-singular M-matrix.
b) There is a positive vector x such that $Ax \gg 0$
c) All principal sub-matrices of A are non-singular M-matrices.
d) All principal minors are positive

Theorem 11: Let A be a matrix with off-diagonal entries are non-positive and A is nonsingular, then the following statements are equivalent: [18]

i) A is an M-matrix.
ii) A is semi positive, that is, there exists $x > 0$ such that $Ax > 0$.
iii) A is monotone, that is, $Ax \geq 0$ implies $x \geq 0$.
iv) $A^{-1} \geq 0$.

Theorem 12: If $A \neq 0$, $x \geq 0$, and $x \neq 0$. If $Ax \geq \alpha x$, for some real α, then $\rho(A) \geq \alpha$ [19].

Theorem 13: If A is an H-matrix, then GJ converges for any initial guess $x^{(0)}$.

Proof: Let A be an H-matrix. Consider of A as $A = T_m - (E_m + F_m)$ for some m. Let $M = T_m = D + R_m$, and $N = E_m + F_m$, where D=diag(A), and $R_m = T_m - D$. Let $H(A)$ be the comparison matrix of A, so that $H(A)$ is an M-matrix.

Note that $H(A) = |D| - |R_m| - |E_m| - |F_m|$. Then $H(A) = M_1 - N_1$ is the generalized Jacobi splitting of $H(A)$, where $M_1 = |D| - |R_m|$ and $N_1 = |E_m| + |F_m|$. Hence by theorem 8 , $\rho(M_1^{-1} N_1) < 1$. Let λ be any eigenvalue of $M^{-1}N$, and let $x \neq 0$ such that $M^{-1}Nx = \lambda x$, that is $Nx = \lambda Mx$, then $|Nx| = |\lambda Mx| \Rightarrow |\lambda||Mx| = |Nx| \leq |N||x|$ implies that

$|\lambda||Dx + R_m x| \leq |N||x|$ Equation (1). Again

$|Dx + R_m x| = |Dx - (-R_m x)| \geq \big||Dx| - |R_m x|\big| \geq |Dx| - |R_m x| = |D||x| - |R_m x|$. Now equation (1) implies that $|\lambda|(|D||x| - |R_m x|) \leq |N||x| \Rightarrow |\lambda|(|D||x| - |R_m||x|) \leq |N||x|$ $\therefore |R_m x| \leq |R_m||x|$. Now $|N| = |E_m + F_m| \leq |E_m| + |F_m| = N_1 \Rightarrow |\lambda|M_1|x| \leq N_1|x|$ equation (2). Since M_1 is an Z-matrix and $H(A) \leq M_1$ so by theorem 10, M_1 is an invertible M-matrix and hence by theorem 11, $M_1^{-1} \geq 0$. equation (2) implies that $|\lambda||x| \leq M_1^{-1} N_1 |x|$. as $M_1^{-1} N_1 \geq 0$, $|x| \geq 0$, and $x \neq 0$, so by theorem 12, $|\lambda| \leq \rho(M_1^{-1} N_1) < 1$. This shows that $\rho(M^{-1}N) < 1$, and GJ method converges.

Theorem 14: ''If A is strictly diagonally dominant matrix, then the refinement of generalized Jacobi method converges for any choice of the initial approximation $x^{(0)}$'' [2].

Theorem 15*:* If A and $2\mathrm{T_m} - \mathrm{A}$ are symmetric and positive definite matrices, then Refinement of Generalized Jacobi is convergent for any initial guess.

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^6 x^{(k-1)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^5]^{-1} T_m^{-1} b$$

$$x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^9 x^{(k-2)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^8]^{-1} T_m^{-1} b$$

$$=\vdots\ x^{(k+1)} = [T_m^{-1}(E_m + E_m^T)]^{3(k+1)} x^{(0)} + [I + \{T_m^{-1}(E_m + E_m^T)\} + \cdots + \{T_m^{-1}(E_m + E_m^T)\}^{3k+2}]^{-1} T_m^{-1} b$$

We have $\rho\big(T_m^{-1}(E_m + E_m^T)\big) < 1$ since A is SPD

$$\Rightarrow \lim_{k\to\infty}[T_m^{-1}(E_m + E_m^T)]^{3(k+1)} x^{(0)} = 0$$

$$\Rightarrow \lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty}[T_m^{-1}(E_m + E_m^T)]^{3(k+1)} x^{(0)} + \sum_{k=0}^{\infty}[T_m^{-1}(E_m + E_m^T))]^k T_m^{-1} b$$

$$= 0 + [I - T_m^{-1}(E_m + E_m^T)]^{-1} T_m^{-1} b \longrightarrow X$$

$\Rightarrow\ \rho(\{T_m^{-1}(E_m + E_m^T)\}^3) < 1 = \rho\big(T_m^{-1}(E_m + E_m^T)\big)^3 < 1$. Therefore, NICA is convergent.

Theorem 21 If A is an M-matrix, and then the generalized Jacobi iterative method is convergent for any initial guess $x^{(0)}$.

Proof: Given A is M-matrix. Let $A = M - N$. $\Rightarrow A = T_m - E_m - F_m \Rightarrow M = T_m$ and $N = E_m + F_m \Rightarrow A \le M \Rightarrow$ by Theorem 3 M is M-matrix.

$\Rightarrow M^{-1} > 0$,

On the other hand $N \ge 0$,

$\therefore A = M - N$ is a regular splitting of the matrix A. Having in minded that $A^{-1} \ge 0$ and Theorem 2 we deduce that $\rho(T_m^{-1}(E_m + F_m)) < 1$

Theorem 22: If **A** is an M-matrix, the new improved classical iterative algorithm (NICA) is convergent for any initial guess $\boldsymbol{x}^{(0)}$.

Proof: It follows from Theorem 21 and the proof of Theorem 17 above.

Theorem 23: If A is an H-matrix, then NICA converges for any initial guess $x^{(0)}$.

Proof: by using theorem 8, $\rho(M_1^{-1}N_1) < 1$. For RGJ we have $\rho((M_1^{-1}N_1)^3) < 1 \Rightarrow \rho(M_1^{-1}N_1)^2 < 1$. It follows from theorem 13 that $|\lambda| \le \rho(M_1^{-1}N_1)^3 < 1$. This shows that $\rho(M^{-1}N)^3 < 1$. Hence NICA converges for any initial guess.

4. RESULTS AND DISCUSSIONS

A number of SOLE's are considered in this research as numerical examples. They include SPD and SDD SOLE's, IDD SOLE's, IDD SOLE's, M-matrix type SOLE's, H-matrix type SOLE's, and SDD that is not PD and SPD SOLE's respectively. In order to show the effectiveness of proposed algorithm over Jacobi (J), Gauss Siedel (GS), Successive Over Relaxation (SOR), Refinement of Jacobi (RJ), Second Refinement of Jacobi (SRJ), Generalized Jacobi (GJ) and Refinement of Generalized Jacobi (RGJ) methods, comparative tables are used. The tables include the factors: (i) spectral radius, (ii) rate of convergence and (iii) number of iterations.

Example 1, [2]: SPD and SDD SOLE's.

$$4x_1 - x_3 - x_4 = 100,$$

$$4x_2 - x_3 - x_4 = 0,$$

$$-x_1 - x_2 + 4x_3 = 100,$$

$$-x_1 - x_2 + 4x_4 = 0,$$

Example 02: IDD SOLE's.

$$4x_1 + x_2 + x_3 + x_5 = 6$$

$$-x_1 - 3x_2 + x_3 + x_4 = 6,$$

$$2x_1 + x_2 + 5x_3 - x_4 - x_5 = 6,$$

$$-x_1 - x_2 - x_3 + 4x_4 = 6,$$

$$2x_2 - x_3 + x_4 + 4x_5 = 6.$$

Example 3, [3]: M-matrix type SOLE's.

$$4x_1 - x_2 - x_4 = 1,$$

$$-x_1 + 4\,x_2 - x_3 - x_5 = 0,$$

$$-x_2 + 4x_3 - \; x_6 = 0,$$

$$-x_1 + 4x_4 - x_5 = 0,$$

$$-x_2 - x_4 + 4x_5 \; - x_6 = 0,$$

$$-x_3 - x_5 + \; 4x_6 = 0.$$

Example 04: H-matrix type SOLE's**.**

$$4x_1 - x_2 - x_3 - x_4 = 1,$$

$$-x_1 + 4x_2 - x_3 - x_4 = 0,$$

$$-x_1 - x_2 + 4x_3 - x_4 = 1,$$

$$-x_1 - x_2 - x_3 + 4x_4 = 0,$$

Example 05, [3]: SDD but not PD and SPD SOLE's.

$$5x_1 + 3\,x_2 + x_3 = 9,$$

$$4x_1 - 6x_2 + x_3 = -1,$$

$$2x_1 + \; x_2 + 4x_3 = 7,$$

Table 1 for the Example 1

Methods	No of Iterations	Spectral Radius	Rate of Convergence
J	22	0.5000	0.3010
GS	12	0.2500	0.6020
SOR	9	0.1300	0.8860
RG	12	0.2500	0.6020
SRJ	9	0.1250	0.9031
GJ	19	0.4396	0.3569
RGJ	10	0.1932	0.7140
NICA	7	0.0850	1.0706

Table 2 for the Example 2

Methods	No of Iterations	Spectral Radius	Rate of Convergence
J	18	0.46911	0.32871
GS	10	0.2752	0.5604
SOR	10	0.2607	0.5839
RG	10	0.2201	0.6574
SRJ	07	0.1032	0.9863
GJ	11	0.3410	0.4672
RGJ	07	0.1163	0.9344
NICA	05	0.0397	1.4012

Table 3 for the Example 3

Methods	No of Iterations	Spectral Radius	Rate of Convergence
J	19	0.6036	0.21921
GS	10	0.3643	0.4385
SOR	08	0.2500	0.6021
RG	10	0.3643	0.4385
SRJ	08	0.2199	0.6578
GJ	12	0.3867	0.4126
RGJ	07	0.1495	0.8254
NICA	05	0.0578	1.2381

Table 4 for the Example 4

Methods	No of Iterations	Spectral Radius	Rate of Convergence
J	34	0.7500	0.1249
GS	20	0.5699	0.2442
SOR	12	0.3312	0.4799
RG	19	0.5625	0.2499
SRJ	13	0.4219	0.3748
GJ	21	0.6048	0.2184
RGJ	12	0.3658	0.4368
NICA	09	0.2212	0.6552

Table 5 for the Example 5

Methods	No of Iterations	Spectral Radius	Rate of Convergence
J	28	0.6227	0.2057
GS	15	0.4258	0.3708
SOR	31	0.6678	0.1754
RG	15	0.3878	0.4114
SRJ	10	0.2415	0.6171
GJ	10	0.0864	1.0635
RGJ	06	0.0518	1.2857
NICA	05	0.0254	1.5952

All results were obtained by using MATLAB Version: 8.2.0.701 (R2013b). Processor was, Intel(R) Core(TM) M-5Y10c CPU @ 0.80GHz 1.00 GHz. The $''\omega''$ in case of SOR is taken optimal. One may try out for m = 2, 3….n-1, where n is the order of matrix, for GJ, RGJ and NICA but in this research paper m=1 is used. The relation used to find rate of convergence is, $R(T_h) = -log10\,[\rho(T_h)]$, where log10 shows common logarithm and $h = j, gs, sor, rj, srj, gj, rgj$ and $nica$.

From numerical experiments one can observe that when SOLE’s is SPD and SDD, IDD, M-Matrix type, H-matrix type and SDD only but not PD and SPD then better method amongst stated methods is NICA, where as the list of second number better methods is SRJ, SRJ, RGJ, SOR and RGJ respectively, as shown in tables 1-5.

5. CONCLUSIONS

The new improved classical iterative algorithm (NICA) minimizes number of iterations almost one - fourth of classical Jacobi, two-fourth of Gauss Seidel or Successive Over Relaxation, three-fourth of Refinement Jacobi and a few number of iterations than other stated methods. The comparison shows that Second Refinement of Jacobi has rapid convergence rate than Successive over Relaxation. The Refinements of Generalized Jacobi method has faster convergence as compared to Gauss Seidel and Jacobi (in most of cases). As mentioned in literature the Gauss seidel has double convergent rate than Jacobi but Successive Over Relaxation is better than Jacobi and Seidel with good choice of parameter ω. From above discussion it can easily deduced that a new improved classical iterative algorithm (NICA) is better than all mentioned methods.

6. CONFLICT OF INTEREST

The authors declare no conflict of interest.

7. REFRENCES

1. A.H Laskar. and S. Behera. "Refinement of iterative methods for the solution of system of linear

equations ax= b." *IOSR Journal of Mathematics (IOSRJM)* 10.3: 70-73 (2014).
2. V.K. Vatti. and G.G. Gonfa. Refinement of generalized Jacobi (RGJ) method for solving system of linear equations. *Int. J. Contemp. Math. Sciences,* 6(3), pp.109-116 (2011).
3. T.K. Eneyew., G. Awgichew, E. Haile, and G.D. Abie. Second Refinement of Jacobi Iterative Method for Solving Linear System of Equations. *IJCSAM (International Journal of Computing Science and Applied Mathematics)*, 5(2), pp.41-47 (2019).
4. F.N. Dafchahi. A new refinement of jacobi method for solution of linear system equations Ax= b. *Int. J. Contemp. Math. Sciences,* 3(17), pp.819-827 (2008).
5. F. Hailu., G.G. Gonfa, and H.M. Chemeda. Second Degree Generalized Successive Over Relaxation Method for Solving System of Linear Equations. *Momona Ethiopian Journal of Science,* 12(1), pp.60-71 (2020).
6. H. Ruan., X. Huang, H. Fu and G. Yang. Jacobi solver: A fast FPGA-based engine system for Jacobi method. Res. *J. Appl. Sci., Eng., Technol.,* 6(23), pp.4459-4463 (2013).
7. M. Tamuli., S. Debnath., A.K. Ray, and S.Majumder. A Review on Jacobi Iterative Solver and its Hardware based performance analysis. In *proceedings of 1st International Conference on Power, Dielectric and Energy Management at NERIST* (pp. 10-11), (2015).
8. D.K. Salkuyeh. Generalized Jacobi and Gauss-Seidel methods for solving linear system of equations. *NUMERICAL MATHEMATICS-ENGLISH SERIES*-, 16(2), p.164 (2007).
9. T.K. Eneyew., G. Awgichew, E. Haile, and G.D. Abie. Second-refinement of Gauss-Seidel iterative method for solving linear system of equations. *Ethiopian Journal of Science and Technology*, 13(1), pp.1-15. (2020).
10. T.K. Eneyew., G. Awgichew, E. Haile, and G.D. Abie. Second Refinement Of Generalized Jacobi Iterative Method For Solving Linear System Of Equations. *Journal of the Nigerian Mathematical Society,* 39(1), pp.117-133. (2020).
11. Y. Saad. *Iterative methods for sparse linear systems.* Society for Industrial and Applied Mathematics (2003).
12. R.S. Varga. *Geršgorin and his circles* (Vol. 36). Springer Science & Business Media (2010).
13. L. Richard and J. Burden. Douglas faires, numerical analysis (2011).
14. R. Bru., C. Corral, I. Gimenez, and Mas, J. Classes of general H-matrices. *Linear algebra and its applications,* 429(10), pp.2358-2366 (2008).
15. B.N. Datta. *Numerical linear algebra and applications* (Vol. 116). Siam (2010).
16. H. Wendland. Multiscale radial basis functions. In Frames and other bases in abstract and function spaces (pp. 265-299). Birkhäuser, Cham (2017).
17. A. Berman and R.J. Plemmons. Nonnegative matrices in the mathematical sciences. Society for Industrial and Applied Mathematics (1994).
18. A. Berman and R.J. Plemmons. Nonnegative matrices in the mathematical sciences. Society for Industrial and Applied Mathematics (1994).
19. R.A. Horn and C.R. Johnson. Matrix Analysis, Cambridge Univ. Press. MR0832183 (1985).