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Research Article

Close Form Solution for Dielectric Cylindrical Shell in Fractional Dimensional Space

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Abstract: We have studied the Laplacian equation in non-integer space which had been previously used to describe complex phenomena in physics and electromagnetism. We have applied this idea to a dielectric cylindrical shell to find the electric potential and field of a dielectric coated cylinder analytically in fractional dimensional space. The problem is derived using Gegenbauer polynomials. This close form gneral solution solved in fractional dimensional space can be applied for various materials of cylindrical shell, outside shell and inside the cylindrical core. The obtained solution is retrieved for integer order by setting the fractional parameter α=3.

Keywords: Fractional dimensional –Space, Laplacian-equation, Dielectric coated cylinder, Electric potential, Analytical Solution, Method of Separation Variables.

1. INTRODUCTION

MThe idea of non-integer space (FD space) is considered to be very useful in various areas of physics and electromagnetism and discussed by many researchers [1-19] and had applied it accordingly. Wilson [3] was the first who applied the non-integer space in quantum field theory.

Further, the non-integer space had been used as a parameter in the Ising limit of quantum field theory [6]. Stillinger [4] provided an axiomatic basis of this concept for the formulation of Schrodinger wave mechanics and Gibbsian statistical mechanics in the α-dimensional space. Svozil and Zeilinger [10] have presented the operationalistic definition of the dimension of space-time which has provided the possibility of experimental determination of the space-time dimension. It has also been stated that the non-integer dimension of space-time is slightly less than 4. In the new era, Gauss law [11] has been formulated in the α-dimensional fractional space. The solutions of electrostatic problems [13-18] have also been investigated in the FD space "($2<\alpha\leq 3$)".

We have extended this problem from J.D. Jackson [13] exercise problem 4.8. The main focus is to use the Laplacian equation to find electric potential and field due to a dielectric cylindrical shell in FD Space. In this paper, the main focus is to use the Laplacian equation to find electric potential and field due to a dielectric cylinder in fractional space. Here the plan of the paper is to be described briefly. In Section 2, we have presented the mathematical model of the boundary value problem where a dielectric cylindrical shell is placed in a uniform electrical field. Then we elaborated potential of dielectric cylindrical shell by solving Laplacian equation to obtain the solution in FD space and constructed the solution for three different regions, namely outside shell, between shell and within the core of the cylinder. Lastly, the unknown constants are determined using boundary conditions and the electric potential and field for the regions in fractional space are derived. Final Section is devoted to our conclusions.

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2. MATHEMATICAL MODEL

We have considered an infinitely long circular cylindrical shell [13] of dielectric constant $\frac{\epsilon}{\epsilon_0}$ for which inside and outside radii are taken to be 'a' and 'b' respectively and are placed in uniform field E_0. The cylinder is oriented with its axis at the right angle to the applied primary field E_0. The medium within the interior cylinder and outside of the exterior cylinder has a dielectric constant of unity.

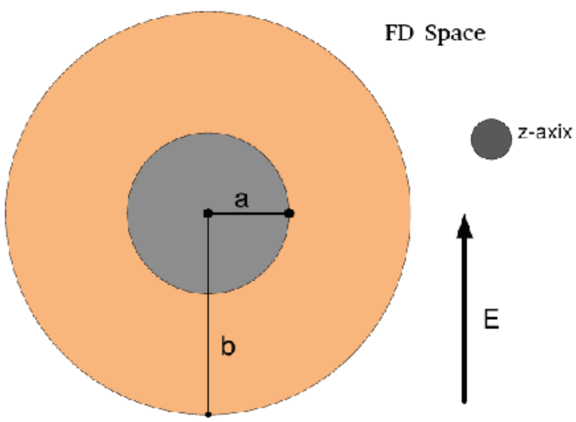

Fig. 1. Dielectric Cylindrical shell Placed in Fractional

We will find the potential and field in fractional space $2 < \alpha \leq 3$ in the three regions. We now employ the cylindrical coordinates (r, θ) for the appropriate solutions.

$$\nabla^2 \Psi(r, \theta) = 0 \tag{1}$$

This is also known as the cylindrical wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial}\right) + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \phi^2} + \frac{\partial^2 \Psi}{\partial z^2} + k^2 \Psi = 0 \tag{2}$$

We will deal this problem in electrostatic and magnetostatics, where $\omega = 0$ so that k=0. Because of the translational symmetry of the problem along the z-axis, Ψ is independent of 'z' and we need only to consider the problem in the (r, θ)-plane. Furthermore, the symmetry in this problem leads us to choose cylindrical coordinates in which Poisson's equation is considered as follows,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0 \tag{3}$$

The Eq(3) is solved by separable method and its possible solutions in the uniform field [17] are $r\cos\theta$ and $r^{-1}\cos\theta$.

The general solution of equation (3) can be expressed as

$$\Psi(r, \phi) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l}\right) P_l(\cos\theta) \tag{4}$$

where, $P_1(\cos\theta) = \cos\theta$.

Eq(3) is solved by separable method in fractional space and suppose

$$\phi(r, \theta) = R(r)\Theta(\theta) \tag{5}$$

The angular and radial differential equations are derived in [18] and [19] and are expressed as,

$$\left[\frac{d^2}{d\theta^2} + (\alpha - 2)\cot\theta \frac{d}{d\theta} + l(l + \alpha - 2)\right]\Theta(\theta) = 0 \tag{6}$$

$$\left[\frac{d^2}{dr^2} + \frac{\alpha - 2}{r}\frac{d}{dr} + \frac{l(l + \alpha - 3)}{r^2}\right]R(r) = 0 \tag{7}$$

The generalized solution of scalar potential of cylinder in fraction space is,

$$\Psi = \sum_{l=0}^{\infty} \left[a_l r^l + b_l r^{-(l+\alpha-3)}\right] P_l^{\alpha/2-1}(\cos\theta) \tag{8}$$

On physical grounds, we can check and contain the derived form of the solution outside and inside the cylindrical region. Outside, we need to have the electric field at infinity, but we certainly do not want the field to diverge. The logarithmic and r^l with $l > 1$ terms diverge as 'r' goes to infinity. Clearly, these terms are unphysical. Therefore, we are interested only in the solution for $l = 1$, where $P_1^{\alpha/2-1}(\cos\theta) = (\alpha - 2)\cos\theta$. Because each region has the same symmetry with respect to the external field, so the expressions of the potentials in each region are expressed as follows,

we find the potetnial outside region,

$$\Psi(r, \phi) = \left(-E_0 r + A_1 r^{-(\alpha-2)}\right)(\alpha - 2)\cos\theta, \quad r > b \tag{9}$$

In between the cylinders:

$$\Psi(r, \phi) = \left(B_1 r + C_1 r^{-(\alpha-2)}\right)(\alpha - 2)\cos\theta, \ a < r < b \tag{10}$$

and In side the cylinder:

$$\Psi_1(r, \phi) = D_1 r(\alpha - 2)\cos\theta, \ 0 < r < a \tag{11}$$

The boundary conditions at $r = a$ and $r = b$ are

$$\frac{\partial\Psi(r,\theta)}{\partial\theta}(b_-) = \frac{\partial\Psi(r,\theta)}{\partial\theta}(b_+) \quad (12)$$

$$\frac{\partial\Psi(r,\theta)}{\partial\theta}(a_-) = \frac{\partial\Psi(r,\theta)}{\partial\theta}(a_+) \quad (13)$$

$$\epsilon\frac{\partial\Psi(r,\theta)}{\partial r}(b_-) = \epsilon_0\frac{\partial\Psi(r,\theta)}{\partial r}(b_+) \quad (14)$$

and

$$\epsilon_0\frac{\partial\Psi(r,\theta)}{\partial r}(a_-) = \epsilon\frac{\partial\Psi(r,\theta)}{\partial r}(a_+) \quad (15)$$

From the above four boundary conditions, we obtain four equations in simplified form such that

$$A = E_0 b^{\alpha-1} + Bb^{\alpha-1} + C \quad (16)$$

$$a_1 A = -E_0 b^{\alpha-1} - \kappa Bb^{\alpha-1} + a_1\kappa C \quad (17)$$

where $a_1 = (\alpha-2)$ and $\kappa = \epsilon/\epsilon_0$

$$D = B + a^{-(\alpha-1)}C \quad (18)$$

$$D = \kappa\left(B - a_1 a^{-(\alpha-1)}C\right) \quad (19)$$

By solving Eq.(16) and Eq.(17) we eliminate the unknown coefficient A and obtain, the following expression

$$C = \frac{((a_1+)E_0 b^{\alpha-1}}{a_1(\kappa-1)} + \frac{(\kappa+a_1)b^{(\alpha-1)}}{a_1(\kappa-1)}B \quad (20)$$

Next we solve Eq.(18) and Eq.(19) to eliminate constant D and get

$$C = \frac{\kappa-1}{\kappa a_1+1}a^{-(\alpha-1)}B \quad (21)$$

Now we compare Eq.(20) and Eq.(21), and easily determine the unknown constant B

$$B = -\frac{(\alpha-1)(\kappa a_1+1)E_0 b^{(\alpha-1)}}{(\kappa+a_1)(\kappa a_1+1)b^{\alpha-1} - a_1(1-\kappa)^2 a^{\alpha-1}} \quad (22)$$

By substitution the value of the constant B in Eq.(22), we find the constant C

$$C = \frac{(\alpha-1)(1-\kappa)E_0 a^{(\alpha-1)}b^{(\alpha-1)}}{(\kappa+a_1)(\kappa a_1+1)b^{\alpha-1} - a_1(1-\kappa)^2 a^{\alpha-1}} \quad (23)$$

From Eq.(18), by substitution of B and C, we find the unknown constant D such that

$$D = -\frac{(\alpha-1)(\alpha-1)E_0 b^{(\alpha-1)}}{(\kappa+a_1)(\kappa a_1+1)b^{\alpha-1} - a_1(1-\kappa)^2 a^{\alpha-1}} \quad (24)$$

Next we substitute the value of B and C in Eq.(16) and obtain the unknown constant A

$$A = E_0 b^{(\alpha-1)} + \frac{(1-\kappa)a^{(\alpha-1)} - (1+\kappa a_1)b^{(\alpha-1)}}{(\kappa+a_1)(\kappa a_1+1)b^{\alpha-1} - a_1(1-\kappa)^2 a^{\alpha-1}} \quad (25)$$

Now we retrieve the exact solution [13] by setting $\alpha = 3$ and $a_1 = (\alpha-2)$, that is given below:

$$B = -\frac{2E_0 b^2(1+\kappa)}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2} \quad (26)$$

By substitution the value of the constant B in Eq.(22), we find the constant C

$$C = \frac{2E_0 a^2 b^2(1-\kappa)}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2} \quad (27)$$

From Eq.(18), by substitution of B and C, we find the unknown constant D such that

$$D = -\frac{4E_0 b^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2} \quad (28)$$

Next we substitute the value of B and C in Eq.(16) and obtain the unknown constant A

$$A = E_0 b^2 + 2E_0 b^2\frac{a^2(1-\kappa) - b^2(1+\kappa)}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2} \quad (29)$$

Special Cases

(1): For a dielectric cylinder, If we the inner radius of the cylinder decreases to zero, that is, $a \to 0$, we get the solution such that In FD Space

$$A = \frac{\kappa-1}{\kappa+1}E_0 b^{\alpha-1} \quad (30)$$

In Integer order

$$A = \frac{\kappa-1}{\kappa+1}E_0 b^2,\ for\ \alpha = 3 \quad (31)$$

In FD Space

$$B = \frac{-(\alpha-1)E_0}{\kappa+1},\ for\ \alpha = 3 \quad (32)$$

In Integer order

$$B = \frac{-2E_0}{\kappa+1},\ for\ \alpha = 3 \quad (33)$$

$$C = 0 \quad (34)$$

In FD Space

$$D = \frac{-(\alpha-1)(\alpha-1)E_0}{(\kappa+1)^2} \quad (35)$$

$$D = \frac{-4E_0}{(\kappa+1)^2},\ for\ \alpha = 3 \quad (36)$$

(2) For the cylindrical cavity, we place the surface of the outer shell at infinity, $b \to \infty$, In this case A is ill-defined, so we would not ignore it.

$$B = \frac{-(\alpha-1)E_0}{\kappa+1} \tag{37}$$

$$B = \frac{-2E_0}{\kappa+1},\ for\ \alpha = 3 \tag{38}$$

In FD Space

$$C = (\alpha - 1)E_0 a^{\alpha-1} \frac{1-\kappa}{(1+\kappa)^2} \tag{39}$$

$$C = 2E_0 a^2 \frac{1-\kappa}{(1+\kappa)^2},\ for\ \alpha = 3 \tag{40}$$

In FD Space ,

$$D = \frac{-(\alpha-1)(\alpha-1)E_0}{(1+\kappa)^2} \tag{41}$$

$$D = \frac{-4E_0}{(1+\kappa)^2},\ for\ \alpha = 3 \tag{42}$$

3. CONCLUSION

In this paper the Laplacian equation has been studied in $\boldsymbol{\alpha}$-dimensional non-integer space. The expressions of potentials and electric fields of the dielectric cylindrical shell are obtained in non-integer space. The classical results are recovered from the investigated solution for $\boldsymbol{\alpha = 3}$. Further, this solution can be applied for various materials. The host medium and core medium can be studied for multiple materials like meta-materials, plasma etc.

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5. CONFLICT OF INTEREST

There is no conflict of interest among the authors.

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