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Research Article

Mathematical Analysis on Spherical Shell of Permeable Material in NID Space

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Abstract: In this paper, we have studied the magnetic shielding effect of a spherical shell analytically in fractional dimensional space (FDS). The Laplacian equation in fractional space predicts the complex phenomena of physics. This is a boundary value problem that has been solved by the separation variable method mathematically by taking low frequency $\omega = 0$. Electric potential is obtained in fractional dimensional space for the three regions, namely outside the spherical shell, between the shell and hollow sphere and inside the sphere. Also, the induced dipole moment has been derived. We obtain a general solution that reduces to the classical results by setting fractional parameter $\alpha = 3$ which takes its value ($2 < \alpha \leq 3$).

Keywords: Variable Method, Magnetic Shielding Effect, Fractional Dimensional Space, Spherical Shell.

1. INTRODUCTION

The novel idea of fractional-dimensional space (FDS) is essential in different disciplines of physics worked by numerous researchers [1-18]. Like the researcher, Wilson [3] has investigated quantum field theory (QFT) in FDS. Furthermore, the FDS can be employed as an indicator in the Ising limit of the QFT [6]. Stillinger [4] has defined an axiomatic basis for this idea for the development of Schrödinger wave mechanics and Gibbsian statistical mechanics in the α-dimensional space. The runtime operational category of space-time dimension shown by Zeilinger and Svozil [10] provides a likelihood of determination of space-time dimension empirically. It is also acknowledged that the fractional dimension of space-time should be less than 4. The α-dimensional fractional space has also been modelled in the last few decades [11]. Moreover, the solution of electro-static problems [13-18], has also been investigated in the FDS ($2 < \alpha \leq 3$).

We have extended the problem of a spherical shell of highly permeable material which is derived by Baleanu *et al.* [17]. We have solved it in fractional dimensional space analytically. The primary aim is to use the Laplacian equation to find electric potential and induced dipole moment in FDS. For the integer order $\alpha = 3$, the original solution is reproduced.

2. MATERIALS AND METHODS

We consider here a spherical shell of permeable material which is placed in fractional space shown in Figure 1**.** We have studied the spherical shell of the inner radius ‘*a*’ and the outer radius ‘*b*’ for the

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phenomenon of magnetic shielding. This problem has been extended from Jackson [13]. The core is made of material of permeability, μ, and placed in a fractional space. B_0 is the uniform magnetic field applied on the surface. We need to discover the fields B and H everywhere in space, but most specifically in the cavity (r < a) as a function of μ. The magnetic field H is determined from a scalar potential H = -∇Ψ, as there are no currents present. Thus, the potential Ψ satisfies the Laplacian in fractional space having fractional α-dimension in "spherical polar coordinate systems" which is described by Baleanu *et al.* [17]:

$$\left(\frac{d^2}{dr^2}+\frac{\alpha-1}{r}\frac{d}{dr}+\frac{1}{r^2}\left[\frac{d^2}{d\theta^2}+(\alpha-2)cot\theta\frac{d}{d\theta}\right]-\frac{1}{r^2sin\theta}\left[\frac{d^2}{d\phi^2}+(\alpha-3)cot\phi\frac{d}{d\phi}\right]\right)\Psi=0. \quad (1)$$

where the fractional parameter α lies in the range (2 < **α** ≤ 3)

In this case, the Laplace equation for the potential independent of angle ϕ can be expressed as:

$$\nabla^2\Psi=\left(\frac{\partial^2}{\partial r^2}+\frac{\alpha-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}\right)\Psi=0 \quad (2)$$

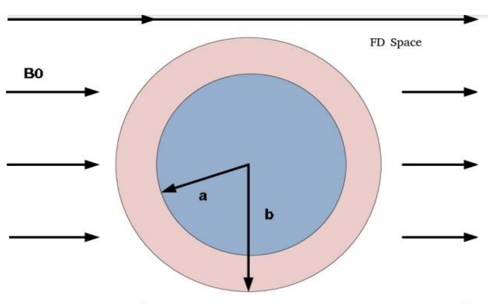

Fig. 1. Spherical Shell of Highly Permeable Material Placed in FDS

Eq (3) is separable and suppose.

$$\Psi(r,\theta)=R(r)\Theta(\theta) \quad (3)$$

The differential equation (3) followed by the published article [17], can be decoupled into two different parts namely angular and radial which are written as:

$$\left[\frac{d^2}{d\theta^2}+(\alpha-2)cot\theta\frac{d}{d\theta}+l(l+\alpha-2)\right]\Theta(\theta)=0 \quad (4)$$

$$\left[\frac{d^2}{dr^2}+\frac{\alpha-1}{r}\frac{d}{dr}+\frac{l(l+\alpha-2)}{r^2}\right]R(r)=0 \quad (5)$$

Therefore, the combined solutions of Ψ (r, θ) in α-dimensional fractional space, can be expressed as

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}\left(a_lr^l+\frac{b_l}{r^{l+\alpha-2}}\right)C_l^{\alpha/2-1}(cos\theta) \quad (6)$$

Here, the unknown constants a_l and b_l can be determined by using the boundary conditions (B.Cs.) on Ψ (r, θ).

We construct here the solution for three different regions by satisfying the B.Cs., at r = a and r = b.

For the outer region r > b, the potential must be of the form,

$$\Psi(r,\theta)=-H_0rcos\theta+\sum_{l=0}^{\infty}\frac{A_l}{r^{l+\alpha-2}}C_l^{\alpha/2-1}(cos\theta) \quad (7)$$

where H = H_0 is the uniform field, at large distance.

For the inner regions, a < r < b the potential can be written as:

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}\left(B_lr^l+\frac{C_l}{r^{l+\alpha-2}}\right)C_l^{\alpha/2-1}(cos\theta) \quad (8)$$

For r<a

$$\Psi(r,\theta)=\sum_{l=0}^{\infty}D_lr^lC_l^{\alpha/2-1}(cos\theta) \quad (9)$$

All coefficients for l ≠ 1 vanish. Then we can construct the solutions for different regions given below:

$$\Psi_e(r,\theta)=\left[-H_0r+Ar^{-(\alpha-1)}\right](\alpha-2)(cos\theta), r>b \quad (10)$$

$$\Psi(r,\theta)=\left[Br+Cr^{-(\alpha-1)}\right](\alpha-2)(cos\theta), a<r<b \quad (11)$$

$$\Psi(r,\theta)=Dr(\alpha-2)(cos\theta), r<a \quad (12)$$

The boundary conditions, at r = a and r = b, are that H_0 and B_r be continuous for l = 1, the coefficients satisfy the four simultaneous equations.

$$\frac{\partial\Psi(r,\theta)}{\partial\theta}\frac{\partial\Psi(r,\theta)}{\partial\theta}(b_-)=\frac{\partial\Psi(r,\theta)}{\partial\theta}(b_+) \quad (13)$$

$$\frac{\partial\Psi(r,\theta)}{\partial\theta}(a_-)=\frac{\partial\Psi(r,\theta)}{\partial\theta}(a_+) \quad (14)$$

$$\mu_1\frac{\partial\Psi(r,\theta)}{\partial r}(b_-)=\mu_0\frac{\partial\Psi(r,\theta)}{\partial r}(b_+) \quad (15)$$

and

$$\mu_0\frac{\partial\Psi(r,\theta)}{\partial r}(a_-)=\mu_1\frac{\partial\Psi(r,\theta)}{\partial r}(a_+) \quad (16)$$

From the above four boundary conditions, we find four simplified equations:

$$A-b^\alpha B-C=a_0 \quad (17)$$

Where, $\alpha_0=H_0b^\alpha$

$$a_1A+\kappa b^\alpha B-a_1\kappa C=-a_0 \quad (18)$$

Where $\alpha_1=\alpha-1$ and $\kappa=\mu/\mu_0$.

$$a^{\alpha}B + C = a^{\alpha}D \tag{19}$$

$$a^{\alpha}\kappa B - a_1\kappa C = a^{\alpha}D \tag{20}$$

By eliminating the unknown constant D from Eq. (19) and Eq. (20), we find

$$C = \frac{(\kappa-1)}{(a_1\kappa+1)}a^{\alpha}B \tag{21}$$

By substituting the value of C in Eq. (18) from Eq. (21), we obtain

$$C = \frac{(a_1+\kappa)}{a_1(\kappa-1)} + \frac{a_0\alpha}{a_1(\kappa-1)} \tag{22}$$

Now we find the value of B by comparing Eq. (21) and Eq. (22).

$$B = \frac{a_0\alpha(a_1\kappa+1)}{(a_1a^{\alpha}(k-1)^2-(a_1+k)(a_1\kappa+1))b^{\alpha}} \tag{23}$$

Similarly,

$$C = \frac{a_0\alpha a^{\alpha}(\kappa-1)}{(a_1a^{\alpha}(k-1)^2-(a_1+k)(a_1\kappa+1))b^{\alpha}} \tag{24}$$

Solving for Constant A, substituting the value of unknown coefficients B and C in Eq. (18), we obtain the simplified coefficient A:

$$A = H_0\frac{[(a_1\kappa+1)(\kappa-1)(b^{\alpha}-a^{\alpha})]}{\left[(a_1+k)(a_1\kappa+1)-a_1\left(\frac{a}{b}\right)^{\alpha}(k-1)^2\right]} \tag{25}$$

Where, $\alpha_1=\alpha-1$ and $\kappa=\mu/\mu_0$.

Finally, we solve for coefficient D by substituting the value of B and C in Eq. (20), we obtain:

$$D = H_0\frac{\alpha(a_1\kappa+1+\kappa-1)}{\left[(a_1+k)(a_1\kappa+1)-a_1\left(\frac{a}{b}\right)^{\alpha}(k-1)^2\right]} \tag{26}$$

Which is simplified as:

$$D = H_0\frac{\alpha^2\kappa}{\left[(a_1+k)(a_1\kappa+1)-a_1\left(\frac{a}{b}\right)^{\alpha}(k-1)^2\right]} \tag{27}$$

To retrieve the results for integer order we set $\boldsymbol{\alpha} = 3$.

Special Case

The potential outside the spherical shell is uniform and the dipole moment equal to the magnitude A. Inside the cavity of highly permeable material, there is a uniform magnetic field parallel to H_0 and equal to magnitude D. For $\mu >> \mu_0$, the dipole moment A and the inner field D become as:

$$A = b^{\alpha}H_0 \tag{28}$$

$$D = \frac{\alpha^2\mu_0}{\mu\left(1-\left(\frac{a}{b}\right)^{\alpha}\right)} \tag{29}$$

3. RESULTS AND DISCUSSION

We have investigated a closed-form solution in non-integer dimensional space (NID) for a spherical shell which is made of magnetic materials. Here, the potential for three regions of the spherical shell has been calculated through induced dipole moment in fractional dimensional space. Its results are very interesting. We find that the field due to the core that is inversely proportional to μ, it means the shielding effect is because of the highly permeable material $\mu/\ \mu_0 \approx 10^3 - 10^6$ that causes enough reduction in the field inside the sphere, although, the spherical shell is thin. Moreover, this general solution can be applied for various materials by replacing permeable materials like lossless metamaterials DNG, ENG, DPS, MNG, ENZ, MNZ, DNZ and plasmas (isotropic, anisotropic, uniaxial, bi-axial, magnetized and un magnetized plasmas). Further, it can be applied for lossy mediums like dry sand, wet sand, water, soil and petroleum etc.

4. CONCLUSIONS

Fractional space plays a key role to describe the complex phenomena of Physics. In this study, the Laplace equation has been analyzed in α-dimensional fractional space (FS). The potential for three regions of the spherical shell is calculated in (FS). Induced dipole moment has been also derived. The shielding effect is because of the highly permeable material $\mu/\mu0 \approx 10^3 - 10^6$ that causes enough reduction in the field inside sphere, although, the spherical shell is thin. A general solution has been investigated in this article that can be applied for different materials inside and outside the spherical shell. For all investigated cases when $\alpha = 3$ the classical results are retrieved.

5. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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