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Research Article

Analytical Solution for Cylindrical Shell of Permeable Material in Fractional Dimensional Space

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Abstract: We have investigated the Laplacian equation in fractional dimensional space (FDS) that is widely used in physics to describe many complex phenomena. Using this concept, we have applied it on a cylindrical shell of permeable material to find the analytical solution of electric potential in FDS. The derivation of this problem is performed by applying Gegenbauer polynomials. The general solution has been obtained in a closed form in the FDS and can be applied to the cylindrical shell for different materials inside the cylinder core and outside the shell**.** By setting the fractional parameter $\alpha = 3$, the derived solution is retrieved for the integer order.

Keywords: General Close Form Solution, Gegenbauer Polynomials, Laplace Equation, Method of Separation Variable.

1. INTRODUCTION

The concept of fractional-dimensional space (FDS) has been applied in many areas of physics for the last many years. It has been investigated and cited by many theoretical physicists [1-19] Wilson [3] has utilized FD space to explain the quantum field theory. Stillinger [4] has formulated Gibbsian statistical mechanics and Schrodinger wave by using this novel idea of the FD space. In the Ising domain [6], the quantum field theory FD space can be utilized as a parameter. Zeilinger and Svozil [10] have elaboratedthe meaning of the dimension of space time by which space-time dimension may be predicted. By using this concept, it has also been estimated that the FD of space-time is approximately less than 4 Gauss law [11] has been derived in α-dimensional fractional space. Various electrostatic issues have been resolved [13-18] in the fractional-dimensional space for $(2 < \alpha \leq 3)$.
This problem has been studied from Jackson [13] for permeable material for the FDS. The main objective of this paper is to evaluate the electric field and potential due to a permeable cylindrical shell by applying the Laplacian equation in FD space. The summary of this paper is described as follows. First of all, we have formulated the boundary value problem. Then in a host medium of uniform electrical field, a permeable cylindrical shell is placed Then, we have evaluated a general solution for the given problem by applying the separation variable method, next we have constructed the solution for the three regions accordingly. Here**,** we apply boundary conditions, and finally**,** we have calculated unknown constants, by using the boundary conditions as well as known values. Now we are able to find the complete solution in form of electric potential due to the cylindrical shell of highly permeable material for fractional dimensional space.

2. MATHEMATICAL MODEL

A cylindrical shell of infinite length filled with permeable material $\mu 1/\mu 0$ having outer and inner radii '*b*' and '*a*' respectively has been considered

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cylinder. We have investigated the potential in NID space ($2 < \alpha \leq 3$), in the three regions. For an appropriate solution the cylindrical coordinates (r, θ) are employed.

$$\nabla^2\Psi(r,\theta) = 0 \quad (1)$$

This is Laplace equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2} + k^2\Psi = 0 \quad (2)$$

As this problem is considered in magneto statics, where $\omega = 0$ it means $k = 0$ and for the symmetry of the problem about the z-axis, Ψ is independent of z, therefore, the above wave equation reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = 0 \quad (3)$$

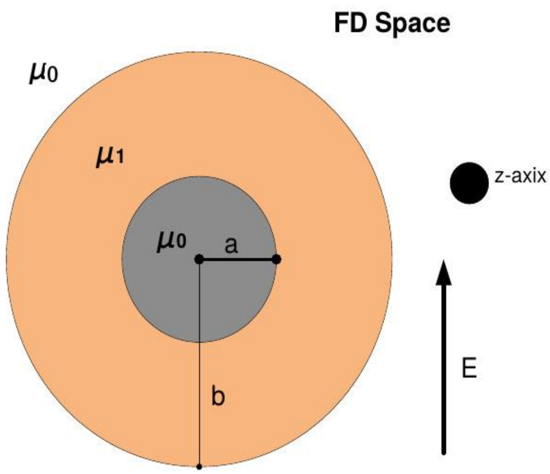

Fig. 1. Cylindrical Shell of Permeable Material Placed in Fractional Space.

Eq. (3) is solved by separable method

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = 0 \quad (4)$$

Suppose the solution is:

$$\Psi(r,\theta) = R(r)\Theta(\theta) \quad (5)$$

The obtained angular and radial part of the differential equations (4) in NID Space by following the previous research work [18, 19], we find:

$$\left[\frac{d^2}{d\theta^2} + (\alpha-2)cot\theta\frac{d}{d\theta} + l(l+\alpha-2)\right]\Theta(\theta) = 0 \quad (6)$$

$$\left[\frac{d^2}{dr^2} + \frac{\alpha-2}{r}\frac{d}{dr} + \frac{l(l+\alpha-3)}{r^2}\right]R(r) = 0 \quad (7)$$

Here, the generalized solution of the scalar potential of cylindrical shell in fractional space followed by Morse and Feshbach [17] can be expressed as:

$$\Psi(r,\theta) = \sum_{l=0}^{\infty}\left[a_l r^l + b_l r^{-(l+\alpha-3)}\right]P_l^{\alpha/2-1}(\cos\theta) \quad (8)$$

For the physical solution, we find a close form solution inside and outside of the cylindrical region. But, we are interested only in the solution for $l = 1$, that is $P^{(\alpha/2-1)}_l(\cos\theta = (\alpha-2)\cos(\theta))$ As there is uniform symmetry with respect to the external field for each region, we find the potential outside region.

$$\Psi(r,\theta) = \left(-E_0 r + A_1 r^{-(\alpha-2)}\right)(\alpha-2)cos\theta, r > b \quad (9)$$

In between the cylinders:

$$\Psi(r,\theta) = \left(B_1 r + C_1 r^{-(\alpha-2)}\right)(\alpha-2)\cos\theta,\ a < r < b \quad (10)$$

and inside the cylinder:

$$\Psi(r,\theta) = D_1 r(\alpha-2)cos\theta, 0 < r < a \quad (11)$$

Here, we have imposed boundary conditions at $r = a$ and $r = b$ These four boundary conditions lead to the following equations in the simplified form such that:

$$A = E_0 b^{\alpha-1} + Bb^{\alpha-1} + C \quad (12)$$

$$a_1 A = -E_0 b^{\alpha-1} - \kappa_1 Bb^{\alpha-1} + a_1\kappa_1 C \quad (13)$$

Where $\boldsymbol{\alpha}_1=(\alpha-2)$ and $\kappa_1=\mu/\mu_0$

$$D = B + a^{-(\alpha-1)}C \quad (14)$$

$$D = \kappa_1\left(B - a_1 a^{-(\alpha-1)}C\right) \quad (15)$$

By solving these equations simultaneously, we find the required unknown constants:

$$B = \frac{-(\alpha-1)(\kappa_1 a_1+1)E_0 b^{(\alpha-1)}}{(\kappa_1+a_1)(\kappa_1 a_1+1)b^{\alpha-1}-a_1} \quad (16)$$

$$C = \frac{(\alpha-1)(1-\kappa_1)E_0 a^{(\alpha-1)}b^{(\alpha-1)}}{(\kappa_1+a_1)(\kappa_1 a_1+1)b^{\alpha-1}-a_1} \quad (17)$$

$$D = \frac{-(\alpha-1)(\alpha-1)E_0 b^{(\alpha-1)}}{(\kappa_1+a_1)(\kappa_1 a_1+1)b^{\alpha-1}-a_1} \quad (18)$$

And

$$A = E_0 b^{(\alpha-1)} + \frac{(1-\kappa_1)a^{(\alpha-1)}-(1+\kappa_1 a_1)b^{(\alpha-1)}}{(\kappa_1+a_1)(\kappa_1 a_1+1)b^{\alpha-1}-a_1} \quad (19)$$

Then we recover the exact solution [13] by setting α = 3 and $\alpha_1 = (\alpha\text{-}2)$ that is given below:

$$B = \frac{-2E_0 b^2(1+\kappa_1)}{b^2} \quad (20)$$

$$C = \frac{2E_0 a^2 b^2(1-\kappa_1)}{b^2} \quad (21)$$

$$D = \frac{-4E_0 b^2}{b^2} \quad (22)$$

Similarly, we find constant A for α=3

$$A = E_0 b^2 + 2E_0 b^2 \frac{a^2(1-\kappa_1)-b^2(1+\kappa_1)}{b^2} \quad (23)$$

Special Cases

For a magnetic cylinder, if we allow a → 0, we obtained the solution such that for the fractional-dimensional space.

$$A = \frac{\kappa_1-1}{\kappa_1+1} E_0 b^{\alpha-1} \quad (24)$$

For Integer order

$$A = \frac{\kappa_1-1}{\kappa_1+1} E_0 b^2, for\ \alpha = 3 \quad (25)$$

In FD Space

$$B = \frac{-(\alpha-1)E_0}{\kappa_1+1},\ for\ \alpha = 3 \quad (26)$$

In Integer order

$$B = \frac{-2E_0}{\kappa_1+1},\ for\ \alpha = 3 \quad (27)$$

$$C = 0 \quad (28)$$

In FD Space

$$D = \frac{-(\alpha-1)(\alpha-1)E_0}{(\kappa_1+1)^2} \quad (29)$$

$$D = \frac{-4E_0}{(\kappa_1+1)^2},\ for\ \alpha = 3 \quad (30)$$

For the cylindrical cavity, if we allow b → ∞, we find

$$B = \frac{-(\alpha-1)E_0}{\kappa_1+1} \quad (31)$$

$$B = \frac{-2E_0}{\kappa_1+1}, for\ \ \alpha = 3 \quad (32)$$

In FD Space

$$C = (\alpha-1)E_0 a^{\alpha-1} \frac{1-\kappa_1}{(1+\kappa_1)^2} \quad (33)$$

$$C = 2E_0 a^2 \frac{1-\kappa_1}{(1+\kappa_1)^2},\ for\ \alpha = 3 \quad (34)$$

In FD Space

$$D = \frac{-(\alpha-1)(\alpha-1)E_0}{(1+\kappa_1)^2} \quad (35)$$

$$D = \frac{-4E_0}{(1+\kappa_1)^2},\ for\ \alpha = 3 \quad (36)$$

3. RESULTS AND DISCUSSION

In this article, we have solved a boundary value problem analytically by using the Laplace equation for NID Space. Here, we have applied four boundary conditions on a cylindrical shell that is made of magnetic material. Then we have calculated an electric potential due to the permeable cylindrical shell. Using the electric potential, we can find electric field. In this article we have extended the classical solution of electric potential in FDS due to the spherical shell of highly permeable material from Jackson [13]. Further, this solution can be applied for various materials. For an example, we can take plasma material as a host medium, also we can fill the cylindrical shell with plasma media like magnetized plasma, anisotropic plasma, isotropic plasma, uniaxial plasma, bi-axial plasma, cold plasma, hot plasma, un magnetized plasma. Similarly, we can also use meta materials as host medium as well as material between the cylindrical shell. The shielding effect due to the highly permeable material causes enough reduction in the field inside cylinder, even if the cylindrical shell is thin. Further, we have also discussed its special cases, (i) If its radius α approaches to zero, we find a solution for a uniform magnetic cylinder, (ii) as we allowed radius *b* approach to infinity, we find a solution for a cylindrical cavity.

4. CONCLUSIONS

In this paper, the analytical solution has been calculated for the permeable cylindrical shell in non-integer dimensional (NID) space. Considering **α** = 3, the classical results are retrieved. The shielding effect due to the highly permeable material causes enough reduction in the field inside cylinder, even if the cylindrical shell is thin. We may apply it further for multiple materials like meta-materials, plasma etc. as the host medium and core medium.

5. CONFLICT OF INTEREST

Authors have no conflict of interest.

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