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Research Article

Numerical Simulation of Nonlinear Equations by Modified Bisection and Regula Falsi Method

Inderjeet and Rashmi Bhardwaj*

Nonlinear Dynamics Research Lab, University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Dwarka, 110078, Delhi, India

Abstract: The study of nonlinear equations and their effective numerical solutions is crucial to mathematical research because nonlinear models are prevalent in nature and require thorough analysis and solution. Many methodologies have been developed to obtain the roots of nonlinear equations, which have significant applications in several areas, especially engineering. However, all of these methods have certain challenges. The development of efficient and effective iterative methods is, therefore, very important and can positively impact the task of finding numerical solutions to many real-world problems. This paper presents a thorough analysis of a numerical approach for solving nonlinear equations using a recently proposed technique, which is a modification of the Regula-Falsi and Bisection numerical methods. The purpose of this work is to provide a novel and effective approach to solving nonlinear equations. The iterative technique for solving nonlinear equations, which has been examined in many scientific and technical domains, is based on the conventional Bisection and Regula-Falsi methods. The proposed approach for finding roots of nonlinear equations achieves second-order convergence. The performance of the newly developed technique was compared with conventional Bisection, Regula-Falsi, Steffensen, and Newton-Raphson methods, and its convergence was validated using several benchmark problems with different iterations. The results showed that, in terms of iterations, the newly developed method performed better than the traditional Bisection, Regula-Falsi, Steffensen, and Newton-Raphson approaches. This supports the credibility of the recently developed method and offers promise for future studies aimed at further refinement. MATLAB R 2021a is used for numerical results. Besides this, the newly developed technique also has certain limitations. For instance, it cannot cover all possible types of nonlinear equations. Further testing on a broader range of functions, particularly those arising from specific scientific and engineering applications, would be valuable. Additionally, our current study focuses on one-dimensional root finding. Extending the approach to systems of nonlinear equations is an important direction for future research.

Keywords: Error Analysis, Convergence Order, Iterations, Numerical Examples.

1. INTRODUCTION

Mathematical equations that model real-world situations are often either linear or nonlinear in nature. The solution to the given problem can be found in the roots of these equations. Since obtaining an accurate solution is essential for problem-solving, an efficient numerical technique for root-finding problems is crucial in mathematical computations. The "root-finding problem" is to identify root of the equation $f(x) = 0$, where f is a function of single variable. Finding root is challenging problem in many fields, such as engineering, chemistry, agriculture, biosciences. This is because formulas for problems involving real-world issues will always include unknown variables. Solving challenges related to identifying an object's equilibrium position, a field's potential surface, and the quantized energy level of a confined structure are pertinent scenarios in the subject of physics [1].

In reality, root-finding problems arise when determining an unknown variable that appears

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* Corresponding Author: Rashmi Bhardwaj <rashmib@ipu.ac.in>

implicitly in scientific or engineering formulas, as reported by Ahmad [2]. The development of optimal eighth-order derivative-free methods for multiple roots of nonlinear equations was discussed by Sharma *et al*. [3]. Several numerical techniques exist for solving root-finding problems, including the Regula-Falsi method, Bisection method, Secant method, Newton-Raphson method, and fixed-point iteration. These methods exhibit various convergence rates, such as linear and quadratic, with higher-order methods converging more quickly, as discussed by Ehiwario and Agnamie [4]. The solution of nonlinear models in engineering using a new sixteenth-order scheme and their basin of attraction was examined by Jamali *et al*. [5].

Numerous studies have been conducted to identify the most effective methods for solving root-finding problems. For example, Frontini and Sormani [6] explored this topic extensively. Similarly, Noor *et al.* [7] analyzed the Secant, Newton-Raphson, and Bisection methods before the research by Ehiwario and Agnamie [4] to determine which approach required fewer iterations when solving a nonlinear equation with a single variable, as noted by Srivastava and Srivastava [8]. Previous research has employed various numerical techniques, such as fixed-point iteration and Regula-Falsi methods, to address root-finding problems, as shown in the studies by Ebelechukwu *et al.* [9] and Issac *et al.* [10]. Additionally, Behl *et al*. [11] discussed a new higher-order optimal derivative-free scheme for multiple roots.

Mathematicians and engineers often struggle to find precise solutions to most real-world problems due to their nonlinear nature [12, 13]. Over the past two decades, several techniques have been proposed or applied in this context [13-19]. Since solving nonlinear equations analytically is highly challenging, iterative procedures based on numerical methods provide the only viable approach to obtaining approximate solutions. Several numerical techniques, including Secant, Bisection, Newton-Raphson, Regula-Falsi, and Muller's methods, are available in the literature for finding approximate roots of nonlinear equations. Cordero *et al.* [20] discussed Steffensen-type methods for solving nonlinear equations.

Many polynomial equations of the form $f(x) = a_0 x^r + a_1 x^{r-1} + a_2 x^{r-2} + \cdots + a_r$ where $a_i's$ are constants, $a_0 \neq 0$ are studied in mathematics. An equation that is transcendental is $f(x) = 0$ if $f(x)$ is constant for certain other functions, such as logarithmic, exponential, trigonometric, etc. A common difficulty in scientific and technological activity is finding root of an equation of the form $f(x) = 0$ [21]. If $f(\omega) = 0$, then any number ω is root of $f(x) = 0$. If $f(x) = (x-\omega)^q g(x)$, where $g(x)$ is bounded at ω and $g(\omega) \neq 0$, then a root of f is said to have multiplicity q. A multiple zero is ω if $q > 1$, and a simple zero if $q = 1$ [22]. By utilizing the classical Regula-Falsi (R-F) approach, Naghipoor *et al.* [23] developed an improved R-F method and demonstrated that the proposed approach was more effective than the classical R-F method. Shaw and Mukhopadhyay [24] introduced an enhanced predictor-corrector form of the R-F approach in their study, showing that it converges significantly faster than the earlier R-F method. Kodnyanko [25] proposed an improved bracketing parabolic method for the numerical solution of nonlinear equations. Additionally, Jamali *et al.* [26] discussed a new two-step optimal approach for solving real-world models and analyzing their dynamics.

To find the roots of nonlinear equations, Parida and Gupta [27] proposed a hybrid approach that combines the standard Regula-Falsi (R-F) method with Newton-like methods. Experiments on multiple examples demonstrate the superiority of this novel strategy over some existing methods for solving similar problems. Li and Chen [28, 29] introduced a technique for finding single roots that integrates the higher-order convergence of the classical R-F approach with specific parameters of the exponential R-F technique. The proposed strategy exhibits good asymptotic quadratic convergence. Qureshi *et al.* [30] discussed the quadratic convergence of iterative algorithms based on the Taylor series for solving nonlinear equations. Numerical analysis research on estimating a single root of nonlinear functions is crucial, as its applications span various fields in both applied and pure sciences. These applications have been discussed within the general framework of nonlinear problems [31-33], such as the nonlinear equation:

$$f(x) = 0 \tag{1}$$

Due to the significance of equation (1), one of the fundamental methodologies, such as the Bisection technique, is used for estimating the root of nonlinear functions.

$$m = \frac{a+b}{2} \quad (2)$$

The technique (2) is a slow yet robust convergence method, known as the Bisection technique [34]. The Bisection technique is guaranteed to converge for a continuous function on an interval $[a, b]$ where $f(a)f(b) < 0$. Alternatively, the Regula-Falsi technique is another root-finding method used for solving nonlinear problems.

$$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \quad (3)$$

The technique (3) is the fast-converging Regula-Falsi method, assessed in comparison to the Bisection technique. Both techniques have linear convergence; however, the Regula-Falsi method occasionally experiences slow convergence. The drawback of the Regula-Falsi technique is mitigated by the Illinois method, as reported by Golbabai and Javidi [35]. Furthermore, a modification of the Newton method was introduced by Chun [36].

Numerous numerical techniques have been proposed, including the quadrature formula, the homotopy perturbation method and its variations, the Taylor series, the divided difference method, the decomposition method, and others [37-41]. Likewise, some two-point algorithms have been developed in the literature to solve nonlinear equations. A similar study, reported by Allame and Azad [42] and Hussain *et al*. [43], integrated the well-known Bisection method, Regula-Falsi method, and Newton-Raphson method to propose a more accurate approach for predicting a single root of nonlinear problems. In this study, a Modified Bracketing technique has been proposed, which combines classical two-point methods. Within the predefined interval, the Modified Bracketing approach performs well in solving nonlinear problems. The proposed method is free of common pitfalls and quickly converges to the root. Moreover, the Modified Bracketing approach is more straightforward and easier to use.

2. METHODOLOGY

In numerical analysis, the Bisection and Regula-Falsi techniques are root-finding methods. Through successive approximation, they use a recursive approach to locate polynomial roots. Let $y = f(x)$ be the graph of an arbitrary function f. Let x_0 and x_1 be an initial approximation of roots of $f(x)$. Let $(x_0, f(x_0))$ and $(x_1, f(x_1))$ be two points on curve $y = f(x)$. Now we draw tangent by joining these two points and x_2 is x-axis intersection of tangent line of function f combining both $(x_0, f(x_0))$ and $(x_1, f(x_1))$ as shown in Figure 1. Equation of line in slope–intercept form joining $(x_0, f(x_0))$ & $(x_1, f(x_1))$ from the Figure 1 is given by:

$$y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0) \quad (4)$$

Equation (4) has a root when y = 0.

$$\rightarrow 0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0) \quad (5)$$

We have solved this equation with respect to *x* as follows:

$$x = x_o - \frac{x_1 - x_0}{f(x_1) - f(x_0)}\, f(x_0) \quad (6)$$

Equation (6) is known as the Regula-Falsi method. The newly created technique is derived by modifying the Regula-Falsi method formula, utilizing both the Bisection and Regula-Falsi techniques. To modify Equation (6), we first use the Bisection method. The Bisection technique is one method for solving the equation $f(x) = 0$. Assume that f is a continuous function defined on the interval $[a, b]$, and that $f(a)f(b) < 0$. Using this technique, the interval $[a, b]$ is divided in half to select a subinterval that satisfies the Intermediate Value Theorem. The selected subinterval is then

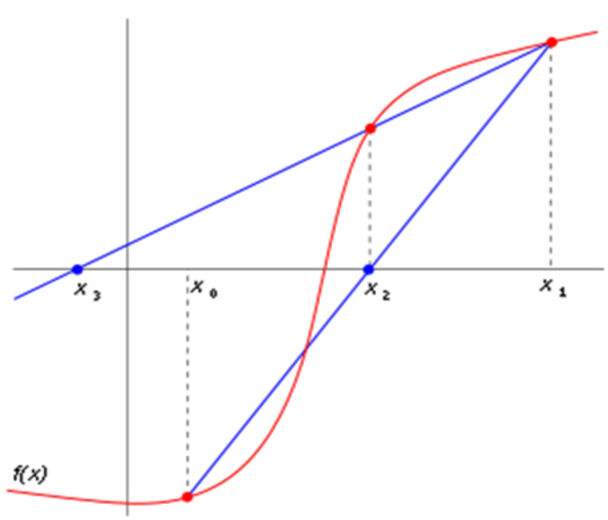

Fig. 1. Geometrical representation of the Regula Falsi method.

extended at both ends and divided in half again. Repeating this process continuously results in an error bound that decreases with each iteration, becoming as small as the interval between successive steps.

So, for first iteration $x_1 = \frac{a+b}{2}$ where $[a, b]$ is an interval in which $f(x)$ has root with $f(a)f(b) < 0$. For better approximation, we refine the interval $[a, b]$ into the smaller subinterval $[a, \frac{a+b}{2}]$ ensuring that $f(a).f(\frac{a+b}{2}) < 0$

$$x_1 = \frac{a+\frac{a+b}{2}}{2} \tag{7}$$

$$x_1 = \frac{3a+b}{4} \tag{8}$$

Also, we know that $f(x_n) = \Delta(x_n) = b - a = h$ with $b = a + f(a)$ and $x_1 = a + 0.3\,f(a)$. Where a is an initial approximation so we can write as $a = x_0$ and $x_1 = x_0 + 0.3\,f(x_0)$. Substituting this expression for $x_1 = x_0 + 0.3\,f(x_0)$ in Equation (6) and solving, we obtain:

$$x = x_0 - \frac{0.3\,[f(x_0)]^2}{f[x_0 + 0.3f(x_0)] - f(x_0)} \tag{9}$$

In general, the recursive formula is:

$$x_{n+1} = x_n - \frac{0.3\,[f(x_n)]^2}{f[x_n + 0.3f(x_n)] - f(x_n)} \quad , n = 0,1,2,3 \tag{10}$$

Hence, Equation (10) represents the proposed modified technique, which integrates the Bisection and Regula-Falsi methods.

3. CONVERGENCE ANALYSIS OF THE MODIFIED TECHNIQUE

Theorem: Let $\alpha \,\varepsilon$ I be simple zero of sufficiently differentiable function $f: I \subseteq R \rightarrow R$ of an open interval I. If x_0 is sufficiently close to α , then technique defined by (8) is of second order and satisfied error equation $e_{n+1} = {e_n}^2[\,0.5\,\frac{f''(a)}{f'(a)} + 0.15\,f''(a)]$

Proof: Let α be simple zero of f, $e_n = x_n - \alpha$. Using Taylor expansion around $x = \alpha$ up - to second order terms and

$f(\alpha) = 0$, we obtained:

$$f(x_n) = f(e_n + \alpha) = f(\alpha) + e_n f'(\alpha) + \frac{{e_n}^2}{2!} f''(\alpha) = e_n f'(\alpha) + \frac{{e_n}^2}{2!} f''(\alpha)$$

$$[f(x_n)]^2 = {e_n}^2\,[f'(a)]^2 + {e_n}^3\,f'(a)f''(a)$$

$$0.3[f(x_n)]^2 = 0.3{e_n}^2[f'(a)]^2 + 0.3{e_n}^3\,f'(a)f''(a)$$

$$x_n + 0.3f(x_n) = e_n + \alpha + 0.3\,e_n f'(\alpha) + 0.3\,\frac{{e_n}^2}{2!}\,f''(\alpha)$$

$$x_n + 0.3f(x_n) = e_n + \alpha + 0.3\,e_n f'(\alpha) + 0.15\,{e_n}^2\,f''(\alpha)$$

$$f[x_n + 0.3f(x_n)] = f[e_n + \alpha + 0.3\,e_n f'(\alpha) + 0.15\,{e_n}^2\,f''(\alpha)]$$

$$f[x_n + 0.3f(x_n)] =$$

$$[e_n + 0.3e_n f'(a) + 0.15{e_n}^2 f''(a)]f'(a) + [e_n + 0.3e_n f'(a) + 0.15{e_n}^2 f''(a)]^2\,\frac{f''(a)}{2!}$$

$$f[x_n + 0.3f(x_n)] = e_n f'(a) + 0.3e_n[f'(a)]^2 + 0.45 \quad {e_n}^2 f'(a)f''(a) + \quad {e_n}^2\,\frac{f''(a)}{2!} \quad + 0.045$$

$e_n{}^2[f'(a)]^2 f''(a)$

$f[x_n + 0.3f(x_n)] - f(x_n) = e_n f'(a) + 0.3e_n[f'(a)]^2 + 0.45e_n{}^2 f'(a)f''(a) + e_n{}^2 \frac{f''(a)}{2!} + 0.045$
$e_n{}^2[f'(a)]^2 f''(a) - e_n f'(a) - e_n{}^2 \frac{f''(a)}{2!}$

$f[x_n + 0.3f(x_n)] - f(x_n) = 0.3e_n[f'(a)]^2 + 0.45e_n{}^2 f'(a)f''(a) + 0.045\, e_n{}^2 [f'(a)]^2 f''(a)$

Substituting values of $x_{n+1}, x_n, 0.3\,[f(x_n)]^2, f[x_n + 0.3f(x_n)] - f(x_n)]$ in (6), we obtained:

$$e_{n+1} + a = e_n + a - \frac{0.3e_n{}^2\,[f'(a)]^2 + 0.3e_n{}^3\,f'(a)f''(a)}{0.3e_n[f'(a)]^2 + 0.45e_n{}^2 f'(a)f''(a) + 0.045\, e_n{}^2\,[f'(a)]^2\, f''(a)}$$

$$e_{n+1} = e_n - \frac{0.3\, e_n[f'(a)]^2[e_n + e_n{}^2\frac{f''(a)}{f'(a)}]}{0.3\, e_n[f'(a)]^2[1 + 1.5e_n\frac{f''(a)}{f'(a)} + 0.15e_n f''(a)}$$

$$e_{n+1} = e_n - \frac{[e_n + e_n{}^2\frac{f''(a)}{f'(a)}]}{[1 + 1.5e_n\frac{f''(a)}{f'(a)} + 0.15e_n f''(a)]}$$

$$e_{n+1} = e_n - [e_n + e_n{}^2\frac{f''(a)}{f'(a)}][1 + 1.5e_n\frac{f''(a)}{f'(a)} + 0.15e_n f''(a)]^{-1}$$

$$e_{n+1} = e_n - [e_n + e_n{}^2\frac{f''(a)}{f'(a)}][1 - \{1.5e_n\frac{f''(a)}{f'(a)} + 0.15e_n f''(a)\} + \{1.5e_n\frac{f''(a)}{f'(a)} + 0.15e_n f''(a)\}^2] + \cdots$$

$$e_{n+1} = e_n - [e_n + e_n{}^2\frac{f''(a)}{f'(a)}]$$

$$[1 - 1.5e_n\frac{f''(a)}{f'(a)} - 0.15e_n f''(a) + 2.25e_n{}^2[\frac{f''(a)}{f'(a)}]^2 + 0.0225\, e_n{}^2[f''(a)]^2 + 0.45\, e_n{}^2\, \frac{[f''(a)]^2}{f'(a)}]$$

$$e_{n+1} = e_n - [e_n - 1.5e_n{}^2\frac{f''(a)}{f'(a)} - 0.15e_n{}^2\, f''(a) + e_n{}^2\frac{f''(a)}{f'(a)} + \ldots]$$

$$e_{n+1} = e_n - e_n + 1.5e_n{}^2\frac{f''(a)}{f'(a)} + 0.15e_n{}^2\, f''(a) - e_n{}^2\frac{f''(a)}{f'(a)} + \ldots$$

$$e_{n+1} = 0.5e_n{}^2\frac{f''(a)}{f'(a)} + 0.15\, e_n{}^2\, f''(a)$$

$$e_{n+1} = e_n{}^2[\,0.5\frac{f''(a)}{f'(a)} + 0.15\, f''(a)]$$

Since the error term satisfies the quadratic form $e_{n+1} = Ce_n{}^2$, the method has second-order convergence. Thus, the equation defined by (10) exhibits quadratic convergence.

4. EFFICIENCY INDEX

The formula for the efficiency index is given by $E = \omega^{1/\alpha}$ where α represents the number of function evaluations required per iteration, and ω denotes the order of convergence of the technique. In this context, the efficiency index of the modified methodology for solving nonlinear equations is 1.189207115. The efficiency indices of the modified methodology and other methods are presented in Table 1 below.

5. RESULTS AND DISCUSSION

For the performance evaluation of the proposed modified method and some existing methods, several nonlinear equations have been considered, as listed in Table 2 along with their test conditions. The well-known Regula Falsi method, Bisection method, Steffensen's method, and Newton-Raphson method are used to compare the proposed modified technique with practical instances of algebraic and transcendental equation analysis. It has been observed that some existing methods have drawbacks and converge slowly, whereas, in certain cases, the proposed method converges more quickly and has no such limitations. By applying these existing approaches to solve nonlinear functions, both the theoretical impact of the proposed method and its experimental validation can be demonstrated. Based on the results, it can be inferred that the modified algorithm operates more efficiently than previously established approaches. Furthermore, the following data representation justifies the number of iterations required by the improved technique in comparison with existing sectioning methods.

Table 1. Efficiency indices for nonlinear equations and systems of the discussed methods.

Methods	Order of Convergence	Total Evaluations in an Iterations	Efficiency Index
Bisection	1	1	1
Regula Falsi	1	2	1
Proposed Modified Method	2	4	1.189207115
Newton Raphson	2	2	1.4142
Steffensen Method	2	3	1.2311

Table 2. Comparison of iteration counts for different root-finding methods.

Function	Root	Interval	Number of Iteration Required				
			BM	RFM	SM	NRM	PM
$e^x - 5x$	0.259171	[0,0.5]	40	9	8	8	7
$x^3 + 4x - 10$	1.55677326439	[1.5,3]	39	26	23	19	13
$\sin x - x + 1$	1.93456	[0,1.5]	39	7	7	7	4
$x \sin x - 1$	1.11415714084	[1,1.5]	35	10	9	7	7
$xe^x - 1$	0.5671432904	[0,1]	37	23	11	8	7
$x^3 + \ln x$	0.7047094901	[0.5,1]	32	14	7	7	7
$x^3 - 9x + 1$	0.11126415759	[0,1]	38	8	5	5	4
$\cos x - 3x + 1$	0.60710164811	[0.60,0.61]	28	13	12	7	5
$x^3 - x - 1$	1.32471795724	[1,2]	36	28	17	11	11
$x^3 - x - 2$	1.5213797	[1,2]	22	15	8	6	4
$x \log x - 1.2$	2.74064609596	[2,3]	38	11	11	7	6

In this study, we present a modified technique for numerically approximating the roots of nonlinear equations. The proposed method enhances the efficiency and convergence rate of the traditional Bisection and Regula-Falsi methods. Theoretical analysis and convergence results demonstrate that the modified method achieves a convergence rate of order 2, which is faster than the traditional Bisection and Regula-Falsi methods.

We proceed with the interpretation of our numerical methods for approximating solutions to nonlinear equations based on the results obtained. The Regula-Falsi, Bisection, Steffensen, Newton-Raphson, and Modified Proposed Methods are used to solve these problems. Table 2 presents the approximations of roots obtained using these methods for nonlinear algebraic and transcendental equations.

The results indicate that, compared to the Regula-Falsi, Bisection, Steffensen, and Newton-Raphson methods, the modified proposed method yields better results while requiring fewer iterations. Additionally, graphical analysis confirms that the proposed method outperforms the Regula-Falsi, Bisection, Steffensen, and Newton-Raphson techniques. Hence, the modified approach presented in this paper offers a powerful and efficient numerical algorithm for solving nonlinear equations.

6. CONCLUSIONS

An iterative, mathematically integrated approach for obtaining the roots of nonlinear equations has been developed and presented in this work. The proposed technique is a modification of the Bisection and Regula-Falsi methods. After comparing the results of this study with those of the Bisection and Regula-Falsi methods, it can be concluded that the proposed modified technique performs quite effectively. This modified method is free of impediments and rapidly converges to the root. Moreover, compared to the Bisection, Regula-Falsi, Steffensen, and Newton-Raphson methods, the proposed modified approach yields superior results in terms of accuracy and iteration count. However, the computational cost of each iteration in the newly suggested method is slightly higher than that of the Bisection and Regula-Falsi methods.

7. ACKNOWLEDGEMENT

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8. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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